

# ON THE EXISTENCE OF A $v_2^{32}$ -SELF MAP ON $M(1,4)$ AT THE PRIME 2

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ABSTRACT. Let  $M(1)$  be the mod 2 Moore spectrum. J.F. Adams proved that  $M(1)$  admits a minimal  $v_1$ -self map  $v_1^4 : \Sigma^8 M(1) \rightarrow M(1)$ . Let  $M(1,4)$  be the cofiber of this self-map. The purpose of this paper is to prove that  $M(1,4)$  admits a minimal  $v_2$ -self map of the form  $v_2^{32} : \Sigma^{192} M(1,4) \rightarrow M(1,4)$ . The existence of this map implies the existence of many 192-periodic families of elements in the stable homotopy groups of spheres.

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## 1. INTRODUCTION

Fix a prime  $p$ . The  $p$ -component of the stable homotopy groups of spheres admits a filtration called the chromatic filtration. Elements in the  $n$ th layer of this filtration fit into infinite  $v_n$ -periodic families. Theoretically, this process is well understood, thanks to the Nilpotence and Periodicity Theorems of Devinatz, Hopkins, and Smith [HS98], [DHS88].

It is difficult in practice, however, to explicitly identify  $v_n$ -periodic elements, and to determine their periods. One useful technique is to inductively form cofiber

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sequences:

$$\begin{aligned}
S &\xrightarrow{p^{i_0}} S \rightarrow M(i_0), \\
\Sigma^{2i_1(p-1)} M(i_0) &\xrightarrow{v_1^{i_1}} M(i_0) \rightarrow M(i_0, i_1), \\
&\vdots \\
\Sigma^{2i_n(p^n-1)} M(i_0, \dots, i_{n-1}) &\xrightarrow{v_n^{i_n}} M(i_0, \dots, i_{n-1}) \rightarrow M(i_0, \dots, i_n).
\end{aligned}$$

The maps  $v_k^i$  are  $v_k$ -self maps. The Periodicity Theorem guarantees their existence for large  $i$ . The reader is warned that there are potentially many non-homotopic  $v_k^i$ -self maps, so the homotopy types of the spectra  $M(i_0, \dots, i_n)$  are not determined merely from the sequence  $(i_0, \dots, i_n)$ .

It is challenging to determine the minimal sequence  $(i_0, i_1, \dots, i_n)$ . This minimal sequence determines the periods of the primary constituents of the  $v_n$ -periodic families in the stable homotopy groups of spheres. We refer the reader to [Rav86, Ch. 5.5], [Rav92], and [Beh07] for a more detailed discussion.

We give a brief synopsis of what is known concerning the minimal sequence of integers  $(i_0, \dots, i_n)$  so that the spectrum  $M(i_0, \dots, i_n)$  exists at a given prime  $p$ . For  $p \geq 3$ , it is known that the complex  $M(1, 1)$  is minimal [Ada66], for  $p \geq 5$ , the complex  $M(1, 1, 1)$  is minimal [Smi70], and for  $p \geq 7$ , the complex  $M(1, 1, 1, 1)$  is minimal [Tod71]. For  $p = 2$ , the complex  $M(1, 4)$  is minimal [Ada66], and for  $p = 3$ , the complex  $M(1, 1, 9)$  is minimal [BP04].

In [DM81], it was argued that the complex  $M(1, 4, 8)$  is minimal at the prime 2, i.e. that there is a  $v_2$ -self map:

$$\Sigma^{48} M(1, 4) \xrightarrow{v_2^8} M(1, 4).$$

The result is incorrect: the image of  $v_2^8$  in the Adams-Novikov spectral sequence for  $tmf$  is not a permanent cycle [HM], [Bau08]. In fact the first multiple of  $v_2$  which is a permanent cycle in this spectral sequence is  $v_2^{32}$ . The purpose of this paper is to prove the following theorem.

**Theorem 1.1.** There is a  $v_2^{32}$ -self map

$$v : \Sigma^{192} M(1, 4) \rightarrow M(1, 4).$$

**Corollary 1.2.** At the prime 2, the complex  $M(1, 4, 32)$  is minimal.

**Remark 1.3.** A  $v_2^{32}$ -self map is, by definition, a map  $v$  whose induced map

$$v_* : K(2)_* M(1, 4) \rightarrow K(2)_* M(1, 4)$$

is given by multiplication by  $v_2^{32}$ . In particular, the map  $v$ , and all of its iterates, must be essential. Since there is a map of ring spectra

$$tmf \rightarrow K(2)$$

under which the periodicity generator  $v_2^{32} \in \pi_{192}(tmf_2)$  maps to  $v_2^{32} \in \pi_{192}K(2)$ , to prove Theorem 1.1, it suffices to prove that there exists a self-map  $v$  such that

$$v_* : tmf_* M(1, 4) \rightarrow tmf_* M(1, 4)$$

is given by multiplication by  $v_2^{32}$ .

**Remark 1.4.** The fourth author reports that methods similar to those described in this paper show that the spectra  $A_1$  and  $M(2, 4)$  also admit  $v_2^{32}$ -self maps. Here,  $A_1$  is a spectrum whose cohomology is a free module of rank 1 over the subalgebra  $A(1)$  of the Steenrod algebra (see [DM81]).

The self-map of Theorem 1.1 produces many  $v_2^{32}$ -periodic infinite families of elements in the stable homotopy groups of spheres. These families are discussed in detail in [HM]. In fact, all of the results of [DM81] and [Mah81] concerning  $v_2$ -periodic families are valid with  $v_2^8$  replaced by  $v_2^{32}$ .

**Organization of the paper.** In Section 2, we reduce Theorem 1.1 to showing that there exists a homotopy element

$$v \in \pi_{192}(M(1, 4) \wedge DM(1))$$

with Hurewitz image  $v_2^{32} \in tmf_{192}(M(1, 4) \wedge DM(1))$ . Here,  $DM(1)$  is the Spanier-Whitehead dual of the spectrum  $M(1)$ .

In Section 3 we construct modified Adams spectral sequences (MASSs) of the form

$$(1.1) \quad \text{Ext}_{A_*}^{s,t}(\mathbb{F}_2, H(1, 4) \otimes DH(1, 4)) \Rightarrow \pi_{t-s}(M(1, 4) \wedge DM(1, 4)),$$

$$(1.2) \quad \text{Ext}_{A_*}^{s,t}(\mathbb{F}_2, H(1, 4) \otimes H_*(X)) \Rightarrow \pi_{t-s}(M(1, 4) \wedge X)$$

where  $A_*$  is the dual Steenrod algebra,  $H(1, 4)$  and  $DH(1, 4)$  are objects in the derived category of  $A_*$ -comodules, and  $\text{Ext}_{A_*}$  is a group of homomorphisms in the derived category. We show that (1.1) is a spectral sequence of algebras, and that (1.2) is a spectral sequence of modules over (1.1).

In Section 4 we prove that there exists an element

$$v_2^8 \in \text{Ext}_{A_*}^{8,56}(\mathbb{F}_2, H(1, 4) \otimes DH(1, 4)).$$

In Section 5, we give a general overview of the theory of generalized Brown-Gitler  $A_*$ -comodules  $M_i(j)$ . We describe a spectral sequence which computes  $\text{Ext}_{A_*}$  in terms of  $\text{Ext}_{A(i)}$  of tensor products of these comodules. The case of interest is where  $i = 2$ , and the spectral sequence is an algebraic version of the  $tmf$ -resolution.

In Section 6 we compute

$$\text{Ext}_{A(2)_*}^{*,*}(H(1, 4) \otimes M_2(1)^{\otimes k})$$

for  $k \leq 3$ .

In Section 7 we establish vanishing lines for the Ext groups appearing in the algebraic  $tmf$ -resolution. These vanishing lines imply that the only targets of a potential differential supported by  $v_2^{32}$  are detected in the algebraic  $tmf$ -resolution by the Ext groups computed in Section 6.

In Section 8, we completely compute the MASS for  $tmf \wedge M(1, 4)$ .

In Section 9, we show that in the MASS for  $M(1, 4) \wedge DM(1, 4)$ , the differential  $d_2(v_2^8)$  is central. This allows us to deduce that  $d_2(v_2^{16}) = 0$ . We then argue that the differential  $d_3(v_2^{16})$  is central, which implies that  $d_3(v_2^{32}) = 0$ . We just need to show that  $v_2^{32}$  is a permanent cycle.

In Section 10, we show that  $\bar{\kappa}^6$  is killed in the  $E_3$ -term of the MASS for  $M(1, 4) \wedge DM(1, 4)$ .

In Section 11, we prove the main theorem. We identify possible targets of  $d_r(v_2^{32})$  in the MASS for  $M(1, 4) \wedge DM(1)$  using the results of Sections 6 and 7, and then eliminate these possibilities using the differentials computed in Section 8 and 10.

**Conventions.** In this paper we shall always be implicitly working in the stable homotopy category localized at the prime 2. All homology and cohomology groups in this paper are implicitly taken with  $\mathbb{F}_2$  coefficients.

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## 2. GENERALIZED MOORE SPECTRA

Let  $M(1)$  be the mod 2 Moore spectrum. There are many  $v_1$ -self-maps

$$\Sigma^8 M(1) \rightarrow M(1),$$

however, low dimensional calculations indicate that there is precisely one with Adams filtration 4. We shall call this map  $v_1^4$ , and its cofiber will be denoted  $M(1, 4)$ .

It is useful to regard the desired self-map  $v$  of Theorem 1.1 as an element of the homotopy group  $\pi_{192}(M(1, 4) \wedge DM(1, 4))$ . The proof of the theorem is simplified by the following splitting result.

**Proposition 2.1** (Davis-Mahowald [DM81, Lem 3.2]). The projection

$$M(1, 4) \wedge DM(1, 4) \rightarrow M(1, 4) \wedge DM(1)$$

is a split surjection.

**Corollary 2.2.** An element  $x \in \pi_k(M(1, 4))$  extends to a self-map

$$\tilde{x} : \Sigma^k M(1, 4) \rightarrow M(1, 4)$$

if and only if  $2x = 0$ .

To prove Theorem 1.1, it therefore suffices to construct an appropriate element  $v' \in \pi_{192}(M(1, 4) \wedge DM(1))$ .

## 3. MODIFIED ADAMS SPECTRAL SEQUENCES

For a graded Hopf algebra  $\Gamma$  over a field  $k$ , let  $\mathcal{D}_\Gamma$  denote the derived category of  $\Gamma$ -comodules. For objects  $M$  and  $N$  of  $\mathcal{D}_\Gamma$ , we define groups

$$\mathrm{Ext}_\Gamma^{s,t}(M, N) = \mathcal{D}_\Gamma(\Sigma^t M, N[s])$$

as a group of maps in the derived category. Here  $\Sigma^t M$  denotes the  $t$ -fold shift with respect to the internal grading of  $M$ , and  $N[s]$  denotes the  $s$ -fold shift with respect to the triangulated structure of  $\mathcal{D}_\Gamma$ . This reduces to the usual definition of  $\mathrm{Ext}_\Gamma$  when  $M$  and  $N$  are  $\Gamma$ -comodules. We shall frequently use the abbreviation

$$\mathrm{Ext}_\Gamma^{*,*}(M) := \mathrm{Ext}_\Gamma^{*,*}(k, M).$$

For a left  $\Gamma$ -comodule  $M$  and a right  $\Gamma$ -comodule  $N$ , let  $C^*(N, \Gamma, M)$  denote the reduced cobar complex with

$$C^s(N, \Gamma, M) = N \otimes \overline{\Gamma}^{\otimes s} \otimes M.$$

Here,  $\overline{\Gamma}$  is the cokernel of the unit  $k \rightarrow \Gamma$ . Then  $C^*(\Gamma, \Gamma, M)$  is an injective resolution for  $M$  in the category of  $\Gamma$ -comodules, and

$$\mathrm{Ext}_{\Gamma}^{s,t}(M) = H^s(C^*(k, \Gamma, M))_t.$$

We refer the reader to [Rav86, Appendix 1] for details.

Let  $A_*$  denote the dual Steenrod algebra. Let  $H(1) = H_*(M(1))$  be the homology of the mod 2 Moore spectrum. There is a triangle in  $\mathcal{D}_{A_*}$ :

$$(3.1) \quad \Sigma \mathbb{F}_2[-1] \xrightarrow{h_0} \mathbb{F}_2 \rightarrow H(1) \rightarrow \Sigma \mathbb{F}_2.$$

Let  $v_1^4 : \Sigma^{12} H(1)[-4] \rightarrow H(1)$  be the unique non-zero element of  $\mathrm{Ext}_{A_*}^{4,12}(H(1), H(1))$ , which detects the  $v_1$ -self map of  $M(1)$  in Adams filtration 4. Let  $H(1, 4)$  denote the cofiber

$$\Sigma^{12} H(1)[-4] \xrightarrow{v_1^4} H(1) \rightarrow H(1, 4) \rightarrow \Sigma^{12} H(1)[-3].$$

Let

$$DM(1, 4) = F(M(1, 4), S) \simeq \Sigma^{-10} M(1, 4)$$

denote the Spanier-Whitehead dual of  $M(1, 4)$ , and let

$$DH(1, 4) = \mathrm{Hom}_{\mathbb{F}_2}(H(1, 4), \mathbb{F}_2) \cong \Sigma^{-13} H(1, 4)[3]$$

denote the corresponding object in  $\mathcal{D}_{A_*}$ .

**Proposition 3.1.** For a finite complex  $X$ , there are *modified Adams spectral sequences* (MASSs) of the form:

$$\begin{aligned} E_2^{s,t}(M(1, 4) \wedge X) &= \mathrm{Ext}_{A_*}^{s,t}(H(1, 4) \otimes H_*(X)) \Rightarrow \pi_{t-s}(M(1, 4) \wedge X), \\ E_2^{s,t}(M(1, 4) \wedge DM(1, 4)) &= \mathrm{Ext}_{A_*}^{s,t}(H(1, 4) \otimes DH(1, 4)) \Rightarrow \pi_{t-s}(M(1, 4) \wedge DM(1, 4)). \end{aligned}$$

*Proof.* Consider the canonical Adams resolution of  $M(1)$ :

$$\begin{array}{ccccccc} M(1) & \xlongequal{\quad} & M(1)_0 & \longleftarrow & M(1)_1 & \longleftarrow & M(1)_2 \longleftarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ & & K(1)_0 & & K(1)_1 & & K(1)_2 \end{array}$$

where

$$\begin{aligned} M(1)_i &= \overline{H}^{\wedge i} \wedge M(1), \\ K(1)_i &= H \wedge \overline{H}^{\wedge i} \wedge M(1). \end{aligned}$$

Here  $H$  denotes the Eilenberg-MacLane spectrum  $H\mathbb{F}_2$ , and  $\overline{H}$  denotes the fiber of the unit  $S \rightarrow H$ . Since the self-map  $v_1^4 : \Sigma^8 M(1) \rightarrow M(1)$  has Adams filtration 4, there exists a lift:

$$\begin{array}{ccc} & & M(1)_4 \\ & \nearrow \widetilde{v_1^4} & \downarrow \\ \Sigma^8 M(1) & \xrightarrow{v_1^4} & M(1) \end{array}$$

The lift  $\widetilde{v}_1^4$  induces a map of Adams resolutions:

$$(3.2) \quad \begin{array}{ccccccc} \Sigma^8 M(1)_0 & \xlongequal{\quad} & \cdots & \xlongequal{\quad} & \Sigma^8 M(1)_0 & \longleftarrow & \Sigma^8 M(1)_1 \longleftarrow \cdots \\ \downarrow v_1^4 & & & & \downarrow (\widetilde{v}_1^4)_0 & & \downarrow (\widetilde{v}_1^4)_1 \\ M(1)_0 & \longleftarrow & \cdots & \longleftarrow & M(1)_4 & \longleftarrow & M(1)_5 \longleftarrow \cdots \end{array}$$

where the maps  $(\widetilde{v}_1^4)_i$  are given by

$$(\widetilde{v}_1^4)_i : \Sigma^8 M(1)_i = \Sigma^8 \overline{H}^{\wedge i} \wedge M(1) \xrightarrow{1 \wedge \widetilde{v}_1^4} \overline{H}^{\wedge i} \wedge \overline{H}^{\wedge 4} \wedge M(1) = M(1)_{i+4}.$$

The mapping cones of the vertical maps of (3.2)

$$\Sigma^8 M(1)_{i-4} \xrightarrow{(\widetilde{v}_1^4)_i} M(1)_i \rightarrow M(1, 4)_i$$

form a resolution:

$$\begin{array}{ccccccc} M(1, 4) & \xlongequal{\quad} & M(1, 4)_0 & \longleftarrow & M(1, 4)_1 & \longleftarrow & M(1, 4)_2 \longleftarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ & & K(1, 4)_0 & & K(1, 4)_1 & & K(1, 4)_2 \end{array}$$

Smashing this resolution with  $X$ , we obtain a spectral sequence

$$(3.3) \quad E_1^{s,t}(M(1, 4) \wedge X) = \pi_{t-s}(K(1, 4)_s \wedge X) \Rightarrow \pi_{t-s}(M(1, 4) \wedge X).$$

By the  $3 \times 3$  Lemma, the cofibers  $K(1, 4)_i$  fit into cofiber sequences

$$(3.4) \quad \Sigma^8 K(1)_{i-4} \xrightarrow{(\overline{v}_1^4)_i} K(1)_i \rightarrow K(1, 4)_i.$$

Here we take  $K(1)_{i-4} = *$  if  $i < 4$ , and  $(\overline{v}_1^4)_i$  is the map induced by smashing  $(\widetilde{v}_1^4)_i$  with  $H$ .

Using the  $A_*$ -comodule structure of  $H(1)$  together with the fact that the composite

$$S^8 \hookrightarrow \Sigma^8 M(1) \xrightarrow{v_1^4} M(1)$$

has Adams filtration 4, one may easily check that the map

$$\Sigma^{12} H(1) = \Sigma^{12} \pi_* K(1)_0 \xrightarrow{(\overline{v}_1^4)_0} \Sigma^4 \pi_* K(1)_4 = C^4(\mathbb{F}_2, A_*, H(1))$$

is injective. It follows that the maps

$$\Sigma^{12} C^{i-4}(\mathbb{F}_2, A_*, H(1)) = \Sigma^{8+i} \pi_* K(1)_{i-4} \xrightarrow{(\overline{v}_1^4)_i} \Sigma^i \pi_* K(1)_i = C^i(\mathbb{F}_2, A_*, H(1))$$

are injective for all  $i$ . We conclude that the cofiber sequences (3.4) give rise to short exact sequences

$$\begin{aligned} 0 \rightarrow \Sigma^{12} C^{i-4}(\mathbb{F}_2, A_*, H(1) \otimes H_* X) & \xrightarrow{(\overline{v}_1^4)_i} C^i(\mathbb{F}_2, A_*, H(1) \otimes H_* X) \\ & \rightarrow \Sigma^i \pi_*(K(1, 4)_i \wedge X) \rightarrow 0. \end{aligned}$$

In the derived category  $D_{A_*}$  we have a map of triangles:

$$\begin{array}{ccccc} \Sigma^{12}C^{*-4}(A_*, A_*, H(1)) & \xrightarrow{(\overline{v_1^4})_*} & C^*(A_*, A_*, H(1)) & \longrightarrow & Q(1, 4)_* \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ \Sigma^{12}H(1)[-4] & \xrightarrow{v_1^4} & H(1) & \longrightarrow & H(1, 4) \end{array}$$

where  $Q(1, 4)_i$  is the cokernel of the inclusion

$$\Sigma^{12}C^{i-4}(A_*, A_*, H(1)) \xrightarrow{(\overline{v_1^4})_i} C^i(A_*, A_*, H(1)).$$

Since we have isomorphisms of cochain complexes

$$\pi_*(K(1, 4)_* \wedge X) \cong \text{Hom}_{A_*}(\mathbb{F}_2, Q(1, 4)_* \otimes H_*X),$$

we deduce that the  $E_2$ -term of the spectral sequence (3.3) is given by

$$E_2^{s,t}(M(1, 4) \wedge X) = \text{Ext}_{A_*}^{s,t}(H(1, 4) \otimes H_*X).$$

Consider the Adams resolution for the Spanier-Whitehead dual  $DM(1)$ :

$$\begin{array}{ccccccc} DM(1) & \xlongequal{\quad} & DM(1)_0 & \longleftarrow & DM(1)_1 & \longleftarrow & DM(1)_2 \longleftarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ & & KD(1)_0 & & KD(1)_1 & & KD(1)_2 \end{array}$$

where

$$\begin{aligned} DM(1)_i &= F(M(1), \overline{H}^{\wedge i}), \\ KD(1)_i &= F(M(1), H \wedge \overline{H}^{\wedge i}). \end{aligned}$$

Define maps  $(\widetilde{Dv_1^4})_i$  to be the composites

$$\begin{aligned} (\widetilde{Dv_1^4})_i : DM(1)_i &= F(M(1), \overline{H}^{\wedge i}) \xrightarrow{u} F(\overline{H}^{\wedge 4} \wedge M(1), \overline{H}^{\wedge i+4}) \\ &\xrightarrow{(\widetilde{v_1^4})^*} F(\Sigma^8 M(1), \overline{H}^{\wedge i+4}) = \Sigma^{-8} DM(1)_{i+4} \end{aligned}$$

where  $u$  is the unit of the adjunction. These maps assemble to give a map of Adams resolutions:

$$\begin{array}{ccccccc} DM(1)_0 & \xlongequal{\quad} & \cdots & \xlongequal{\quad} & DM(1)_0 & \longleftarrow & M(1)_1 \longleftarrow \cdots \\ \widetilde{Dv_1^4} \downarrow & & & & (\widetilde{Dv_1^4})_0 \downarrow & & (\widetilde{Dv_1^4})_1 \downarrow \\ \Sigma^{-8} DM(1)_0 & \longleftarrow & \cdots & \longleftarrow & \Sigma^{-8} DM(1)_4 & \longleftarrow & \Sigma^{-8} DM(1)_5 \longleftarrow \cdots \end{array}$$

Letting  $DM(1, 4)_i$  denote the homotopy fibers of the vertical maps of (3):

$$DM(1, 4)_i \rightarrow DM(1)_i \xrightarrow{(\widetilde{Dv_1^4})_i} \Sigma^{-8} DM(1)_{i+4},$$

we obtain a modified Adams resolution of  $DM(1, 4)$ :

$$\begin{array}{ccccccc} DM(1, 4) & \xlongequal{\quad} & DM(1, 4)_{-4} & \longleftarrow & DM(1, 4)_{-3} & \longleftarrow & DM(1, 4)_{-2} \longleftarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ & & KD(1, 4)_{-4} & & KD(1, 4)_{-3} & & KD(1, 4)_{-2} \end{array}$$

and a corresponding modified Adams spectral sequence

$$E_2^{s,t}(DM(1, 4)) = \text{Ext}_{A_*}^{s,t}(DH(1, 4)) \Rightarrow \pi_{t-s}(DM(1, 4)).$$

By taking iterated mapping cylinders, we may assume that the maps

$$\begin{aligned} M(1, 4)_{i+1} &\rightarrow M(1, 4)_i, \\ DM(1, 4)_{i+1} &\rightarrow DM(1, 4)_i \end{aligned}$$

are inclusions of subcomplexes. Taking the smash product of resolutions [BMMS86, Ch. IV, Def. 4.2]

$$\{(M(1, 4) \wedge DM(1, 4))_i\} = \{M(1, 4)_i\} \wedge \{DM(1, 4)_i\}$$

gives the spectral sequence

$$\text{Ext}_{A_*}^{s,t}(H(1, 4) \otimes DH(1, 4)) \Rightarrow \pi_{t-s}(M(1, 4) \wedge DM(1, 4)).$$

□

**Proposition 3.2.** The spectral sequence  $\{E_r(M(1, 4) \wedge DM(1, 4))\}$  is a spectral sequence of algebras, and the spectral sequence  $\{E_r(M(1, 4) \wedge X)\}$  is a spectral sequence of modules over  $\{E_r(M(1, 4) \wedge DM(1, 4))\}$ .

*Proof.* The canonical Adams resolution for the sphere spectrum is given by  $\{\overline{H}^{\wedge i}\}$ . The canonical evaluation maps

$$\begin{aligned} DM(1, 4)_i \wedge M(1, 4)_j &= \\ (DM(1)_i \times_{(Dv_1^4)_i} (\Sigma^{-8} DM(1)_{i+4})^I) \wedge (M(1)_j \cup_{(v_1^4)_j} C\Sigma^8 M(1)_{j-4}) &\rightarrow \overline{H}^{\wedge i+j} \end{aligned}$$

induce maps of modified Adams resolutions

$$\begin{aligned} \{(M(1, 4) \wedge DM(1, 4))_i\} \wedge \{(M(1, 4) \wedge DM(1, 4))_i\} & \\ = \{M(1, 4)_i\} \wedge \{(DM(1, 4) \wedge M(1, 4))_i\} \wedge \{DM(1, 4)_i\} & \\ \rightarrow \{M(1, 4)_i\} \wedge \{\overline{H}^{\wedge i}\} \wedge \{DM(1, 4)_i\} & \\ = \{(M(1, 4) \wedge DM(1, 4))_i\}, & \end{aligned}$$

$$\begin{aligned} \{(M(1, 4) \wedge DM(1, 4))_i\} \wedge \{M(1, 4)_i \wedge X\} & \\ = \{M(1, 4)_i\} \wedge \{(DM(1, 4) \wedge M(1, 4))_i\} \wedge \{\overline{H}^{\wedge i} \wedge X\} & \\ \rightarrow \{M(1, 4)_i\} \wedge \{\overline{H}^{\wedge i}\} \wedge \{\overline{H}^{\wedge i} \wedge X\} & \\ = \{M(1, 4)_i \wedge X\}. & \end{aligned}$$

These maps induce the desired pairings on the corresponding MASSs.

□



4.  $v_2^8$ -PERIODICITY IN  $\text{Ext}_{A_*}$ 

A similar (but easier) argument to Proposition 2.1 proves the following lemma.

**Lemma 4.1.** The morphism

$$H(1, 4) \wedge DH(1, 4) \rightarrow H(1, 4) \wedge DH(1)$$

is a split surjection.

**Corollary 4.2.** An element  $x \in \text{Ext}_{A_*}^{s,t}(H(1, 4))$  lifts to give an element  $\tilde{x}$  in  $\text{Ext}_{A_*}(H(1, 4) \otimes DH(1, 4))$  if and only if  $h_0x = 0$ .

A computation of  $\text{Ext}_{A(2)_*}(H(1, 4))$  appears in Figure 8.1. Note that it is  $v_2^8$ -periodic.

**Proposition 4.3.** There exists an element

$$\tilde{v}_2^8 \in \text{Ext}_{A_*}^{8,56}(H(1, 4) \otimes DH(1, 4))$$

which maps to the element  $v_2^8 \in \text{Ext}_{A(2)_*}^{8,56}(H(1, 4))$  under the composite

$$\text{Ext}_{A_*}^{*,*}(H(1, 4) \otimes DH(1, 4)) \rightarrow \text{Ext}_{A_*}^{*,*}(H(1, 4)) \rightarrow \text{Ext}_{A(2)_*}(H(1, 4)).$$

*Proof.* In the May spectral sequence for  $\text{Ext}_{A(2)_*}(\mathbb{F}_2)$ , the element  $v_2^8$  is detected by  $b_{3,0}^4$ . Using Nakamura's formula [Nak72], and the calculations of [Tan70], we see that in the May spectral sequence for  $\text{Ext}_{A_*}(\mathbb{F}_2)$ , there are differentials:

$$\begin{aligned} d_8(b_{3,0}^4) &= b_{2,0}^4 h_5, \\ d_4(b_{2,0}^2 h_5) &= h_0^4 h_3 h_5. \end{aligned}$$

In the May spectral sequence,  $v_1^4$  multiplication corresponds to multiplication by  $b_{2,0}^2$ . It follows that an element of  $\text{Ext}_{A_*}^{8,56}(H(1, 4))$  which maps to

$$v_2^8 \in \text{Ext}_{A(2)_*}^{8,56}(H(1, 4))$$

must have image  $h_0^3 h_3 h_5$  under the composite

$$\text{Ext}_{A_*}^{8,56}(H(1, 4)) \xrightarrow{\delta_{v_1^4}} \text{Ext}_{A_*}^{5,44}(H(1)) \xrightarrow{\delta_{v_0}} \text{Ext}_{A_*}^{5,43}(\mathbb{F}_2).$$

Since the element  $h_0^3 h_3 h_5 \in \text{Ext}_{A_*}^{5,43}(\mathbb{F}_2)$  is killed by  $h_0$  multiplication, it lifts to an element  $h_0^3 h_3 h_5[1] \in \text{Ext}_{A_*}^{5,44}(H(1))$ . Consider the exact sequence

$$\text{Ext}_{A_*}^{8,56}(H(1)) \rightarrow \text{Ext}_{A_*}^{8,56}(H(1, 4)) \rightarrow \text{Ext}_{A_*}^{5,44}(H(1)) \xrightarrow{v_1^4} \text{Ext}_{A_*}^{9,56}(H(1)).$$

A computer calculation of  $\text{Ext}_{A_*}^{*,*}(H(1))$  using Bruner's programs [Bru93] reveals that:

- (1)  $\text{Ext}_{A_*}^{9,56}(H(1)) = 0$ ,
- (2) Every element  $x \in \text{Ext}_{A_*}^{5,44}(H(1))$  satisfies  $h_0x = 0$ ,
- (3) Every element  $y \in \text{Ext}_{A_*}^{9,57}(H(1))$  satisfies  $y = h_0z$  for some  $z \in \text{Ext}_{A_*}^{8,56}(H(1))$ .

These three facts allow us to deduce that there exists an element  $w \in \text{Ext}_{A_*}^{8,56}(H(1, 4))$  which maps to  $h_0^3 h_3 h_5[1]$ , and for which we have  $h_0w = 0$ . By Corollary 4.2, the element  $w$  lifts to the desired element  $\tilde{v}_2^8$  in  $\text{Ext}_{A_*}^{8,56}(H(1, 4) \otimes DH(1, 4))$ .  $\square$

We shall abusively refer to the element  $\tilde{v}_2^8 \in \text{Ext}_{A_*}^{8,56}(H(1, 4) \otimes DH(1, 4))$  as  $v_2^8$ .

## 5. BROWN-GITLER COMODULES

**Definitions.** Let  $A(i)_*$  denote the quotient of the dual Steenrod algebra dual to the subalgebra  $A(i)$  of the Steenrod algebra. There is an isomorphism

$$A(i)_* \cong \mathbb{F}_p[\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3, \dots, \bar{\xi}_{i+1}] / (\bar{\xi}_1^{2^{i+1}}, \bar{\xi}_2^{2^i}, \bar{\xi}_3^{2^{i-1}}, \dots, \bar{\xi}_{i+1}^2).$$

Here,  $\bar{\xi}_i$  denotes the conjugate of  $\xi_i$ . We define a filtration on  $A_*$  which induces a filtration on the  $A_*$ -subcomodule

$$(A//A(i))_* = A_* \square_{A(i)_*} \mathbb{F}_2 \cong \mathbb{F}_2[\bar{\xi}_1^{2^{i+1}}, \bar{\xi}_2^{2^i}, \dots, \bar{\xi}_{i+1}^2, \bar{\xi}_{i+2}, \dots].$$

Our filtration is an increasing filtration of algebras given on generators by  $|\bar{\xi}_j| = 2^{j-1}$ . In particular, every element of  $(A//A(i))_*$  has filtration divisible by  $2^{i+1}$ . The Brown-Gitler comodule  $N_i(j)$  is the subspace of  $(A//A(i))_*$  spanned by all elements of filtration less than or equal to  $2^{i+1}j$ . Using the coproduct formula

$$(5.1) \quad \psi(\bar{\xi}_k) = \sum_{k_1+k_2=k} \bar{\xi}_{k_1} \otimes \bar{\xi}_{k_2}^{2^{k_1}},$$

the submodule  $N_i(j)$  is easily seen to be an  $A_*$ -subcomodule. Thus we have an increasing sequence of  $A_*$ -comodules:

$$\mathbb{F}_2 \cong N_i(0) \subset N_i(1) \subset N_i(2) \subset \dots \subset (A//A(i))_*.$$

Define a map of ungraded rings

$$\phi_i : (A//A(i))_* \rightarrow (A//A(i-1))_*$$

whose effect on generators is given by:

$$\phi_i(\bar{\xi}_k^{2^l}) = \begin{cases} \bar{\xi}_{k-1}^{2^l}, & k > 1, \\ 1, & k = 1. \end{cases}$$

**Lemma 5.1.** The map  $\phi_i$  is a map of ungraded  $A(i)_*$ -comodules.

*Proof.* As an  $A(i)_*$ -comodule algebra,  $(A//A(i))_*$  is generated by the elements  $\{\bar{\xi}_1^{2^{i+1}}, \bar{\xi}_2^{2^i}, \dots\}$ . It therefore suffices to check that  $\phi_i$  commutes with the coaction on these generators. This is easily checked using the coproduct formula (5.1) and the relations in  $A(i)_*$ .  $\square$

Let  $M_i(j)$  denote the subspace of  $(A//A(i))_*$  spanned by the monomials of filtration exactly  $2^{i+1}j$ .

**Lemma 5.2.** The map  $\phi_i$  maps the subspace  $M_i(j)$  isomorphically onto the  $A_*$ -subcomodule  $N_{i-1}(j) \subset (A//A(i-1))_*$ .

*Proof.* The subspace of  $M_i(j)$  spanned by monomials of the form  $\bar{\xi}_1^{2^{i+1}} s x$  where  $x$  is a monomial involving  $\bar{\xi}_k^{2^l}$  for  $k > 1$  is mapped isomorphically onto the subspace  $M_{i-1}(j-s) \subset N_{i-1}(j)$ .  $\square$

Using Lemma 5.1, we have the following corollaries.

**Corollary 5.3.** The subspace  $M_i(j) \subset (A//A(i))_*$  is an  $A(i)_*$ -subcomodule.

**Corollary 5.4.** There is an isomorphism of (graded)  $A(i)_*$ -comodules

$$M_i(j) \cong \Sigma^{2^{i+1}j} N_{i-1}(j).$$

**Corollary 5.5.** There is a splitting of  $A(i)_*$ -comodules

$$(A//A(i))_* \cong \bigoplus_{j \geq 0} M_i(j).$$

**Remark 5.6.** The comodule  $N_{-1}(j)$  (respectively  $N_0(j)$ ,  $N_1(j)$ ) is isomorphic as an  $A_*$ -comodule to the homology of the  $j$ th  $\mathbb{Z}/2$  (respectively integral, *bo*) Brown-Gitler spectrum. It is not known in general if the comodules  $N_i(j)$  are realizable for  $i > 1$ .

**Algebraic resolutions.** We now describe an algebraic analog of an Adams resolution. For  $i = -1$  (respectively  $i = 0, 1, 2$ ) this algebraic resolution will correspond to the  $H\mathbb{F}_2$  (respectively  $H\mathbb{Z}$ , *bo*, *tmf*) Adams resolution.

Let  $X$  be an object of the derived category  $\mathcal{D}_{A_*}$ . We define  $T_i(X)^\bullet$  to be the following cosimplicial object.

$$(A//A(i))_* \otimes X \begin{array}{c} \xrightarrow{u \otimes 1} \\ \xrightarrow{1 \otimes u} \end{array} (A//A(i))_*^{\otimes 2} \otimes X \begin{array}{c} \xrightarrow{-u \otimes 1 \otimes 1} \\ \xrightarrow{-1 \otimes u \otimes 1} \\ \xrightarrow{-1 \otimes 1 \otimes u} \end{array} (A//A(i))_*^{\otimes 3} \otimes X \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array}$$

Here,  $u$  is the unit

$$\mathbb{F}_2 \rightarrow (A//A(i))_*.$$

Since  $(A//A(i))_*$  is an algebra, the canonical map

$$X \rightarrow \text{Tot}(T^i(X)^\bullet)$$

is a quasi-isomorphism (see, for instance, [Wei94, Prop. 8.6.8]). We therefore have a Bousfield-Kan spectral sequence

$$(5.2) \quad E_1^{s,t,n} = \text{Ext}_{A_*}^{s,t}((A//A(i))_* \otimes \overline{(A//A(i))_*}^{\otimes n} \otimes X[-n]) \Rightarrow \text{Ext}_{A_*}^{s,t}(X).$$

where

$$\overline{(A//A(i))_*} = \text{coker} \left( \mathbb{F}_2 \xrightarrow{u} (A//A(i))_* \right).$$

The  $E_1$ -term can be simplified using a change of rings isomorphism, together with the splitting of Corollary 5.5:

$$\begin{aligned} E_1^{s,t,n} &= \text{Ext}_{A_*}^{s,t}((A//A(i))_* \otimes \overline{(A//A(i))_*}^{\otimes n} \otimes X[-n]) \\ &\cong \text{Ext}_{A(i)_*}^{s,t}(\overline{(A//A(i))_*}^{\otimes n} \otimes X[-n]) \\ &\cong \bigoplus_{j_1, \dots, j_n \geq 1} \text{Ext}_{A(i)_*}^{s,t}(M_i(j_1) \otimes \cdots \otimes M_i(j_n) \otimes X[-n]). \end{aligned}$$

We shall call this spectral sequence (5.2) the  $A//A(i)$ -resolution for  $X$ . In this paper we are only be interested in the case where  $i = 2$ . In this case, we shall refer to the  $A//A(2)$ -resolution as the *algebraic tmf-resolution*.

**Lemma 5.7.** If  $R$  is a monoid in the derived category  $\mathcal{D}_{A_*}$ , then the  $A//A(i)$ -resolution for  $R$  is a spectral sequence of algebras. If  $M$  is an  $R$ -module, then the  $A//A(i)$ -resolution for  $M$  is a spectral sequence of modules over the  $A//A(i)$ -resolution for  $R$ .

## 6. EXT COMPUTATIONS

In this section we describe  $\text{Ext}_{A(2)*}^{s,t}(M)$  for various objects  $M \in \mathcal{D}_{A(2)*}$ . We first explain the computations, and then describe the methodology used to produce these computations. Charts displaying these Ext groups can be found in the following figures:

- Figure 6.1:  $\text{Ext}_{A(2)*}^{*,*}(\mathbb{F}_2)$  and  $\text{Ext}_{A(2)*}^{*,*}(M_2(1))$ ,
- Figure 6.2:  $\text{Ext}_{A(2)*}^{*,*}(M_2(1)^{\otimes 2})$  and  $\text{Ext}_{A(2)*}^{*,*}(M_2(1)^{\otimes 3})$ ,
- Figure 6.3:  $\text{Ext}_{A(2)*}^{*,*}(M_2(1) \otimes H(1))$  and  $\text{Ext}_{A(2)*}^{*,*}(M_2(1)^{\otimes 2} \otimes H(1))$ ,
- Figure 6.4:  $\text{Ext}_{A(2)*}^{*,*}(M_2(1)^{\otimes 3} \otimes H(1))$  and  $\text{Ext}_{A(2)*}^{*,*}(M_2(1) \otimes H(1, 4))$ ,
- Figure 6.5:  $\text{Ext}_{A(2)*}^{*,*}(M_2(1)^{\otimes 2} \otimes H(1, 4))$  and  $\text{Ext}_{A(2)*}^{*,*}(M_2(1)^{\otimes 3} \otimes H(1, 4))$ .

In each of these charts, the indexing has been modified to put the bottom generator of  $M_2(1)^{\otimes k}$  in internal degree 0. The meaning of the notation in each of these charts is explained below.

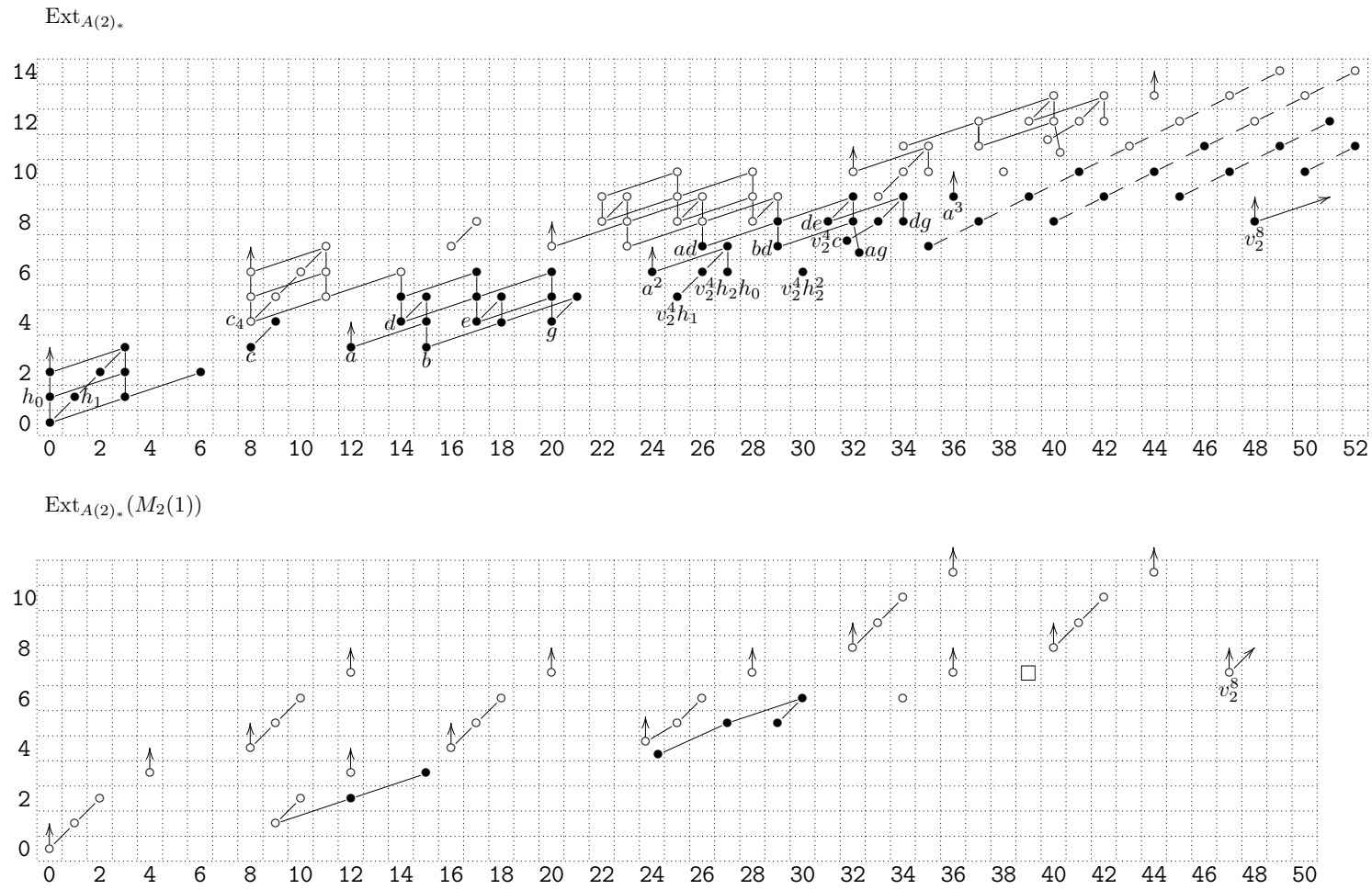
**$\text{Ext}_{A(2)*}(\mathbb{F}_2)$ .** All of the elements are  $c_4 = v_1^4$ -periodic, and  $v_2^8$ -periodic. Exactly one  $v_1^4$  multiple of each element is displayed with the  $\bullet$  replaced by a  $\circ$ . Observe the wedge pattern beginning in  $t - s = 35$ . This pattern is infinite, propagated horizontally by  $h_{2,1}$ -multiplication and vertically by  $v_1$ -multiplication. Here,  $h_{2,1}$  is the name of the generator in the May spectral sequence of bidegree  $(t-s, s) = (5, 1)$ , and  $h_{2,1}^4 = g$ .

**$\text{Ext}_{A(2)*}(M_2(1)^{\otimes k})$ , for  $k = 1, 2, 3$ .** Every element is  $v_2^8$ -periodic. However, unlike  $\text{Ext}_{A(2)*}(\mathbb{F}_2)$ , not every element of these Ext groups is  $v_1^4$ -periodic. Rather, it is the case that either an element  $x \in \text{Ext}_{A(2)*}(M_2(1)^{\otimes k})$  satisfies  $v_1^4 x = 0$ , or it is  $v_1^4$ -periodic. Each of the  $v_1^4$ -periodic elements fit into families which look like shifted and truncated copies of  $\text{Ext}_{A(1)*}(\mathbb{F}_2)$ , and are labeled with a  $\circ$ . We have only included the beginning of these  $v_1^4$ -periodic patterns in the chart. The other generators are labeled with a  $\bullet$ . A  $\square$  indicates a polynomial algebra  $\mathbb{F}_2[h_{2,1}]$ .

**$\text{Ext}_{A(2)*}(M_2(1)^{\otimes k} \otimes H(1))$ , for  $k = 1, 2, 3$ .** The notation in these charts is identical to that in the charts for  $\text{Ext}_{A(2)*}(M_2(1)^{\otimes k})$ , with the exception that the  $v_1^4$ -periodic patterns are truncated shifted copies of  $\text{Ext}_{A(1)*}(H(1))$ .

**$\text{Ext}_{A(2)*}(M_2(1)^{\otimes k} \otimes H(1, 4))$ , for  $k = 1, 2, 3$ .** Because we have taken the cofiber of  $v_1^4$ , none of the elements are  $v_1^4$  periodic in these charts. The generators of the first  $v_2^8$ -periodic pattern are denoted with a  $\bullet$  or a  $\square$ , where again a  $\square$  denotes a polynomial algebra on  $h_{2,1}$ . In these charts, however, it is not the case that every element is  $v_2^8$ -periodic: some elements in the first lightening flash in the 0-stem fail to be  $v_2^8$ -periodic. We have conveyed this information by displaying the elements in the next  $v_2^8$ -pattern with  $\circ$ . With the exception of these first few generators, all of the other generators are  $v_2^8$ -periodic.

FIGURE 6.1.



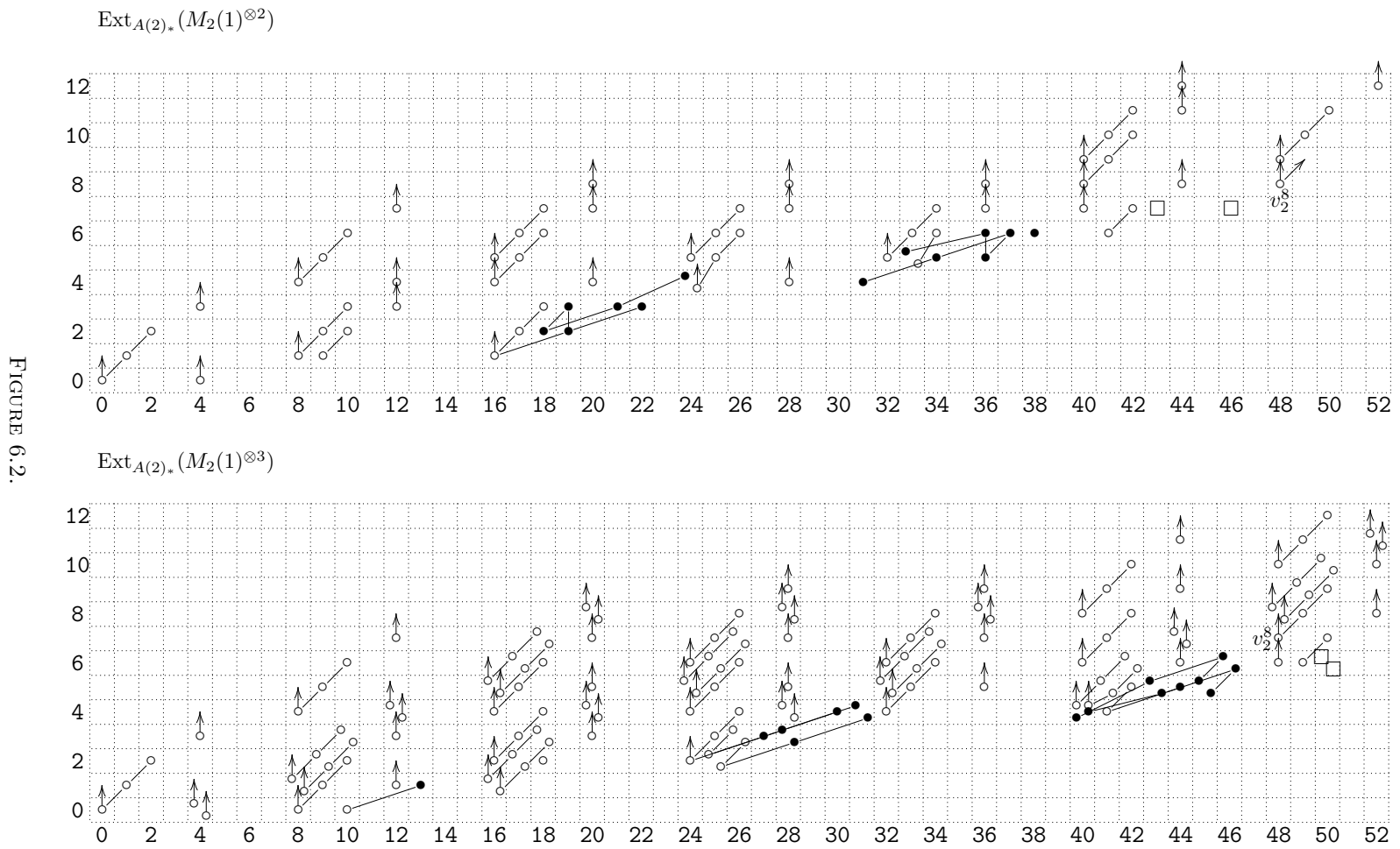
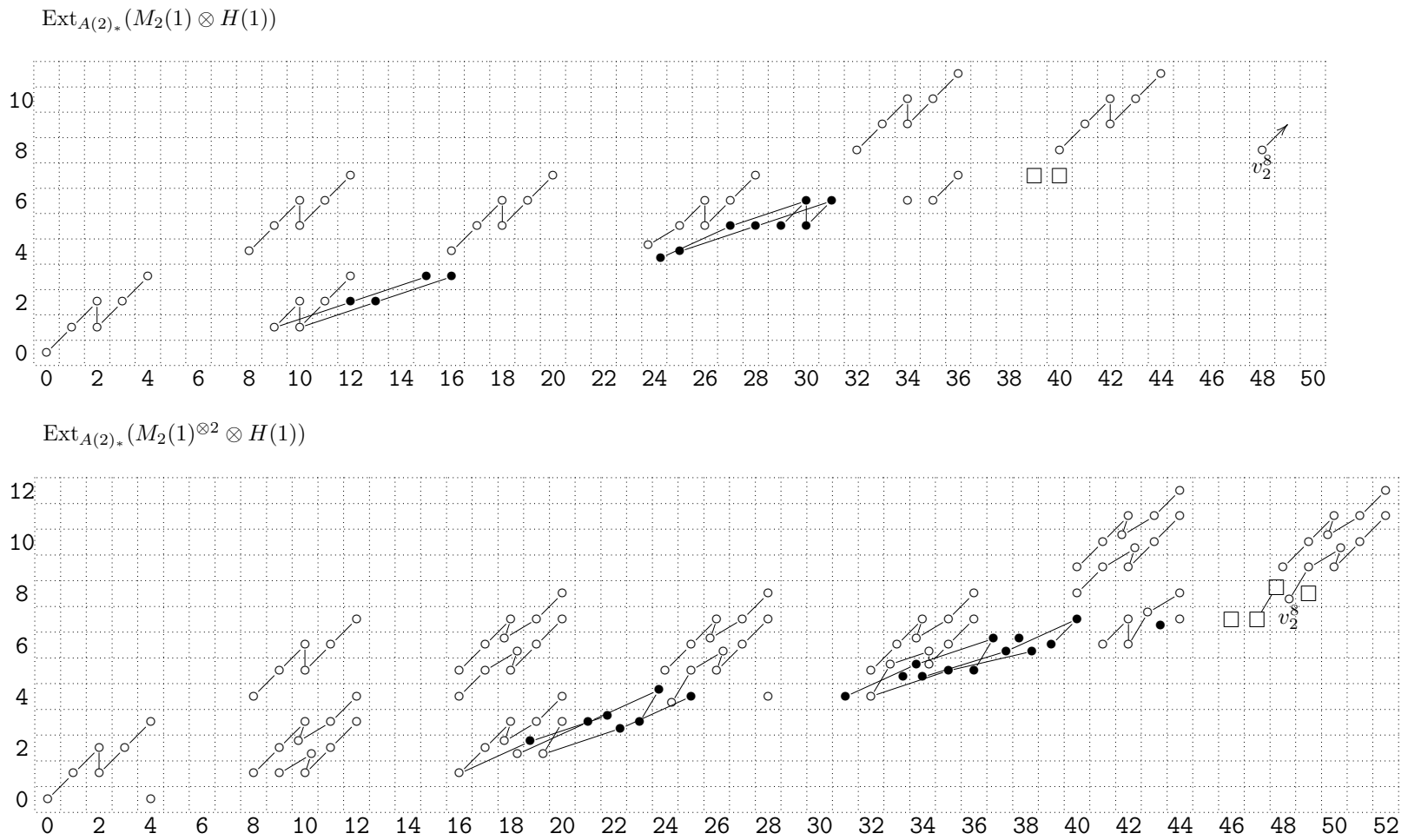


FIGURE 6.3.



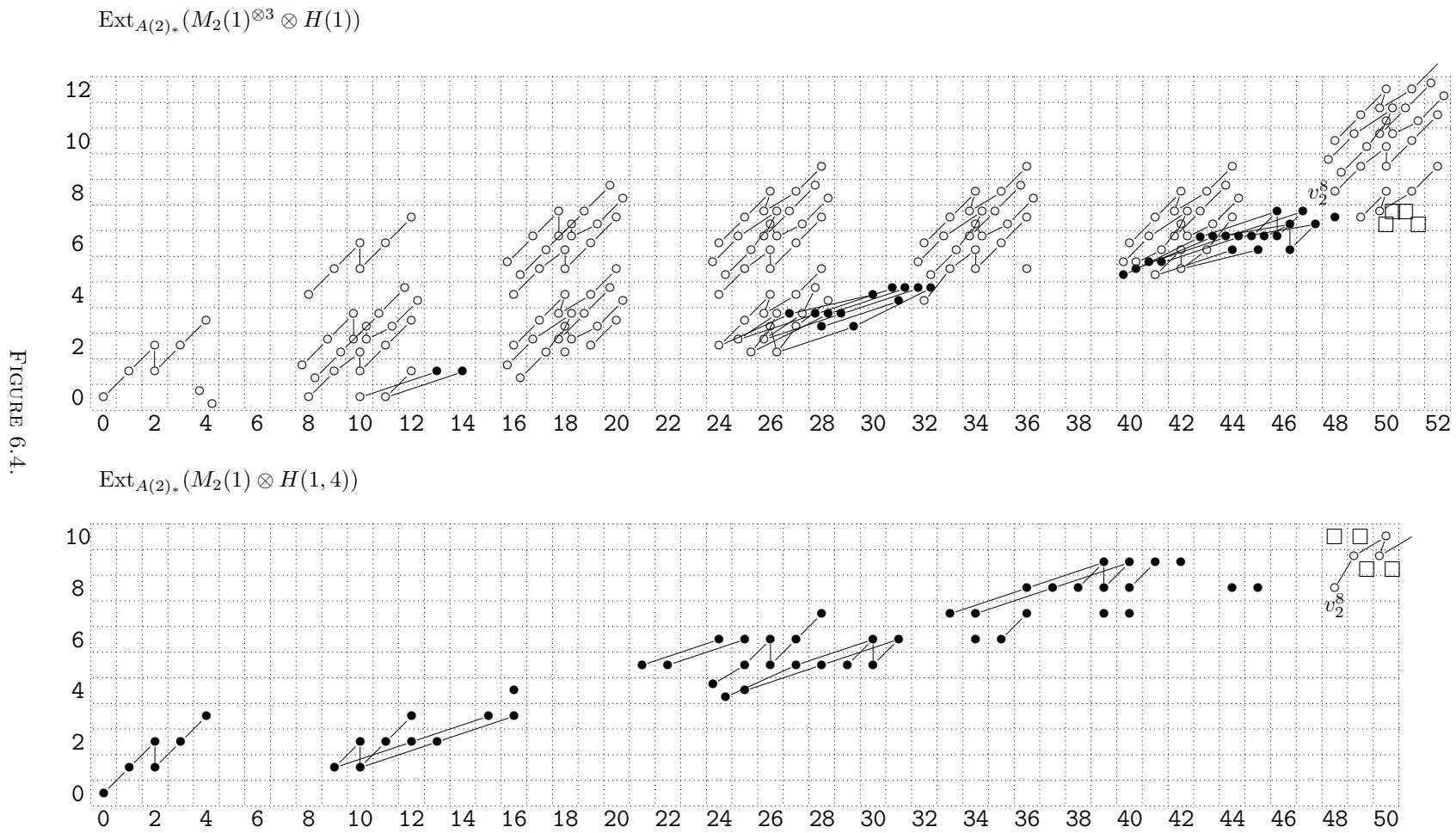


FIGURE 6.4.



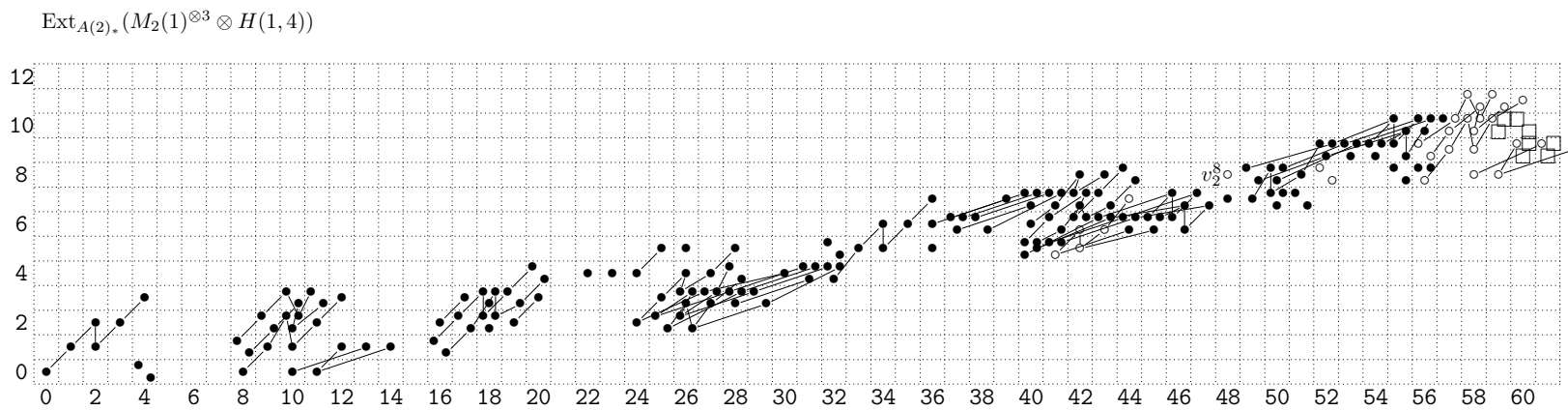
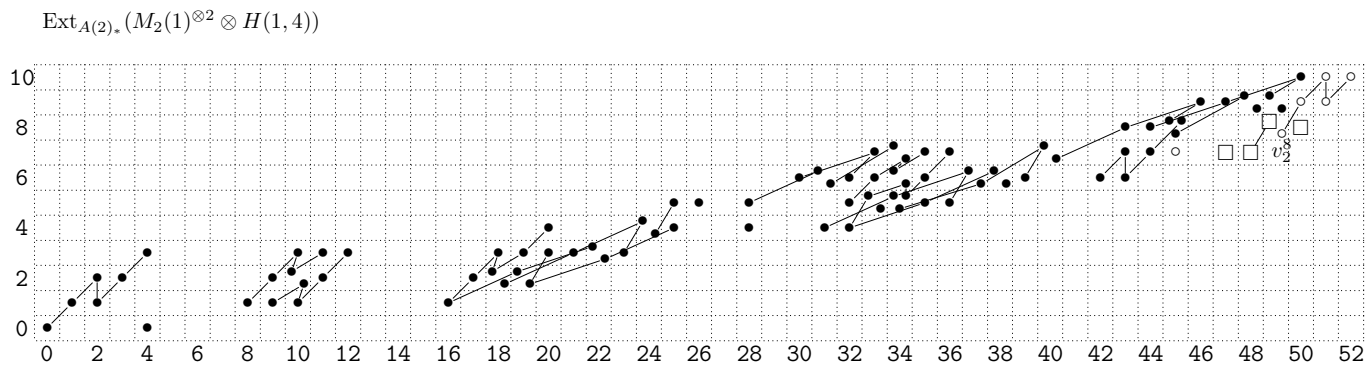
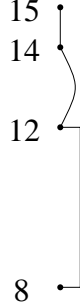


FIGURE 6.5.

**Methodology.** We explain how these charts were produced. The computation of  $\text{Ext}_{A(2)_*}(\mathbb{F}_2)$  is well-known (see, for instance, [DM82]). The  $A(2)_*$  comodule  $M_2(1)$  can be described by the following diagram of generators.



Here, the dual action of the Steenrod algebra is encoded with a straight line denoting  $Sq_*$ , a curved line denoting  $Sq_*^2$ , and the bracket denoting  $Sq_*^4$ . A computation of  $\text{Ext}_{A(2)_*}(M_2(1))$  can be found in [DM82]. The computation of  $\text{Ext}_{A(2)_*}(M_2(1)^{\otimes 2})$  was obtained from  $\text{Ext}_{A(2)_*}(M_2(1))$  by inductively working up the skeletal filtration of the second factor of  $M_2(1)$ . The computation of  $\text{Ext}_{A(2)_*}(M_2(1)^{\otimes 3})$  was then obtained from  $\text{Ext}_{A(2)_*}(M_2(1)^{\otimes 2})$  by inductively working up the skeletal filtration of the third factor of  $M_2(1)$ . Along the way, because  $H(1)$  occurs as a subcomodule of  $M_2(1)$ , we have computed  $\text{Ext}_{A(2)_*}(M_2(1)^{\otimes k} \otimes H(1))$  for  $k = 1, 2$ . We then use the long exact sequence induced by the triangle (3.1) to obtain  $\text{Ext}_{A(2)_*}(M_2(1)^{\otimes 3} \otimes H(1))$ .

Each of these manual computations was independently verified by R.R. Bruner's computer program for computing  $\text{Ext}$  [Bru93]. This computer program constructs minimal resolutions of modules over the subalgebra  $A(2)$ . We also used the computer program to gain complete understanding of  $v_1^4$ -periodicity in these  $\text{Ext}$  groups, as we now explain. Note that there is an element

$$v_1 \in \text{Ext}_{A(2)_*}^{1,3}(H(1) \otimes H_*C\eta)$$

where  $C\eta$  is the cofiber of  $\eta \in \pi_1^s$ . We used Bruner's programs to compute minimal resolutions for

$$\text{Ext}_{A(2)_*}(M_2(1)^{\otimes k} \otimes H(1) \otimes H_*C\eta), \quad k = 1, 2, 3,$$

and read off all of the  $v_1$ -multiplicative structure in these  $\text{Ext}$  groups from the minimal resolutions. We then used an  $\eta$ -Bockstein spectral sequence to recover the  $v_1^4$ -multiplicative structure on

$$\text{Ext}_{A(2)_*}(M_2(1)^{\otimes k} \otimes H(1)), \quad k = 1, 2, 3.$$

From this, the computation of

$$\text{Ext}_{A(2)_*}(M_2(1)^{\otimes k} \otimes H(1, 4)), \quad k = 1, 2, 3$$

was easily determined by the long exact sequence arising from the triangle (3).

## 7. REDUCING THE COMPUTATION TO $M_2(1)^{\otimes k}$ FOR $k \leq 3$

**Inductive Short Exact Sequences.** We will construct some short exact sequences that relate the various Brown-Gitler comodules  $N_1(j)$ . We have an isomorphism

$$(A(2) // A(1))_* \cong \Lambda[\bar{\xi}_1^4, \bar{\xi}_2^2, \bar{\xi}_3].$$

Observe that there is an isomorphism of  $\mathbb{F}_2$ -vector spaces

$$\tau : (A//A(1))_* \xrightarrow{\cong} (A//A(2))_* \otimes (A(2)//A(1))_*$$

given on the monomial basis by

$$\tau(\bar{\xi}_1^{8i_1+4\epsilon_1} \bar{\xi}_2^{4i_2+2\epsilon_2} \bar{\xi}_3^{2i_3+\epsilon_3} \bar{\xi}_4^{i_4} \dots) = \bar{\xi}_1^{8i_1} \bar{\xi}_2^{4i_2} \bar{\xi}_3^{2i_3} \bar{\xi}_4^{i_4} \dots \otimes \bar{\xi}_1^{4\epsilon_1} \bar{\xi}_2^{2\epsilon_2} \bar{\xi}_3^{\epsilon_3}$$

for  $i_j \geq 0$  and  $\epsilon_j = 0, 1$ . The map  $\tau$  is *not* an isomorphism of  $A(2)_*$ -comodules. For instance, in  $(A//A(1))_*$  we have the coaction

$$\psi(\bar{\xi}_1^4 \bar{\xi}_2^2) = \bar{\xi}_1^4 \bar{\xi}_2^2 \otimes 1 + \bar{\xi}_1^4 \otimes \bar{\xi}_2^2 + \bar{\xi}_2^2 \otimes \bar{\xi}_1^4 + 1 \otimes \bar{\xi}_1^4 \bar{\xi}_2^2 + \bar{\xi}_1^6 \otimes \bar{\xi}_1^4 + \bar{\xi}_1^2 \otimes \bar{\xi}_1^8$$

whereas in  $(A//A(2))_* \otimes (A//A(1))_*$  we have

$$\psi(1 \otimes \bar{\xi}_1^4 \bar{\xi}_2^2) = \bar{\xi}_1^4 \bar{\xi}_2^2 \otimes 1 \otimes 1 + \bar{\xi}_1^4 \otimes 1 \otimes \bar{\xi}_2^2 + \bar{\xi}_2^2 \otimes 1 \otimes \bar{\xi}_1^4 + 1 \otimes 1 \otimes \bar{\xi}_1^4 \bar{\xi}_2^2 + \bar{\xi}_1^6 \otimes 1 \otimes \bar{\xi}_1^4.$$

However, there is a decreasing filtration

$$(A//A(1))_* = F^0(A//A(1))_* \supset F^1(A//A(1))_* \supset \dots$$

of  $A(2)_*$ -comodules such that  $\tau$  induces an isomorphism of the associated graded  $A(2)_*$ -comodules

$$\tau : E^0(A//A(1))_* \xrightarrow{\cong} (A//A(2))_* \otimes (A(2)//A(1))_*.$$

The decreasing filtration is given as follows: under the isomorphism

$$(A//A(2))_* \cong \bigoplus_k M_2(k)$$

of  $A(2)_*$ -comodules given by Corollary 5.5, we define

$$F^j(A//A(1))_* := \tau^{-1} \left( \left( \bigoplus_{k=j}^{\infty} M_2(k) \right) \otimes (A(2)//A(1))_* \right).$$

Using the coproduct formula (5.1) this is easily verified to be a decreasing filtration by  $A(2)_*$ -comodules — the coaction preserves or raises the filtration.

Consider the quotients

$$Q^j(A//A(1))_* := (A//A(1))_* / F^{j+1}(A//A(1))_*.$$

The map  $\tau$  induces isomorphisms of  $\mathbb{F}_2$ -vector spaces

$$\tau : Q^j(A//A(1))_* \xrightarrow{\cong} N_2(j) \otimes (A(2)//A(1))_*.$$

Furthermore, the filtration  $\{F^k(A//A(1))_*\}$  projects to a finite decreasing filtration of  $Q^j(A//A(1))_*$  by  $A(2)_*$ -comodules, such that  $\tau$  induces an isomorphism of associated graded  $A(2)_*$ -comodules

$$(7.1) \quad \tau : E^0 Q^j(A//A(1))_* \xrightarrow{\cong} N_2(j) \otimes (A(2)//A(1))_*.$$

**Lemma 7.1.** There is a short exact sequence of  $A(2)_*$ -comodules:

$$0 \rightarrow \Sigma^{8j} N_1(j) \otimes N_1(1) \rightarrow N_1(2j+1) \rightarrow Q^{j-1}(A//A(1))_* \rightarrow 0.$$

**Lemma 7.2.** There is an exact sequence of  $A(2)_*$ -comodules:

$$0 \rightarrow \Sigma^{8j} N_1(j) \rightarrow N_1(2j) \rightarrow Q^{j-1}(A//A(1))_* \rightarrow \Sigma^{8j+9} N_1(j-1) \rightarrow 0.$$

*Proof of Lemma 7.1.* Since the elements of  $(A(2)//A(1))_*$  have Brown-Gitler filtration at most 12, the image of the composite

$$N_2(j-1) \otimes (A(2)//A(1))_* \hookrightarrow (A//A(2))_* \otimes (A(2)//A(1))_* \xrightarrow{\tau^{-1}} (A//A(1))_*$$

lies in  $N_1(2j+1)$ , giving a surjection of  $A(2)_*$ -comodules

$$\rho : N_1(2j+1) \twoheadrightarrow Q^{j-1}(A//A(1))_*.$$

As  $\mathbb{F}_2$ -vector spaces, we have

$$\tau(N_1(2j+1)) = N_2(j-1) \otimes (A(2)//A(1))_* \oplus M_2(j) \otimes N_1(1)$$

where the Brown-Gitler comodule  $N_1(1)$  is identified as the  $A(2)_*$ -subcomodule

$$N_1(1) = \mathbb{F}_2\{1, \bar{\xi}_1^4, \bar{\xi}_2^2, \bar{\xi}_3\} \subset (A(2)//A(1))_*.$$

We deduce that the kernel of  $\rho$  is

$$M_2(j) \otimes N_1(1) \cong \Sigma^{8j} N_1(j) \otimes N_1(1).$$

□

*Proof of Lemma 7.2.* As an  $\mathbb{F}_2$ -vector space, the image of  $N_1(2j)$  in  $(A//A(2))_* \otimes (A(2)//A(1))_*$  under the isomorphism  $\tau$  is given by

$$\tau(N_1(2j)) \cong \left( \begin{array}{c} N_2(j-2) \otimes (A(2)//A(1))_* \\ \oplus \\ M_2(j-1) \otimes \mathbb{F}_2\{1, \bar{\xi}_1^4, \bar{\xi}_2^2, \bar{\xi}_3, \bar{\xi}_1^4 \bar{\xi}_2^2, \bar{\xi}_1^4 \bar{\xi}_3, \bar{\xi}_2^2 \bar{\xi}_3\} \\ \oplus \\ M_2(j) \otimes \mathbb{F}_2\{1\}. \end{array} \right)$$

Thus, at least on the level of  $\mathbb{F}_2$ -vector spaces, we have an exact sequence

$$0 \rightarrow M_2(j) \otimes \mathbb{F}_2\{1\} \xrightarrow{\alpha} N_1(2j) \xrightarrow{\beta} Q^{j-1}(A//A(1))_* \xrightarrow{\gamma} M_2(j-1) \otimes \mathbb{F}_2\{\bar{\xi}_1^4 \bar{\xi}_2^2 \bar{\xi}_3\} \rightarrow 0$$

We just need to prove that these are maps of  $A(2)_*$ -comodules. The map  $\gamma$  is clearly a map of  $A(2)_*$ -comodules. We have the following diagram of inclusions of  $A(2)_*$ -comodules.

$$(7.2) \quad \begin{array}{ccccc} M_2(j) \otimes \mathbb{F}_2\{1\} & \xhookrightarrow{\alpha} & N_1(2j) & \xhookrightarrow{\delta} & Q^j(A//A(1))_* \\ \downarrow & & \downarrow & \nearrow & \\ (A//A(2))_* & \hookrightarrow & (A//A(1))_* & & \end{array}$$

In particular, the map  $\alpha$  is a map of  $A(2)_*$ -comodules. Let  $K$  be the cokernel of  $\alpha$ . Then we get an induced map of short exact sequences of  $A(2)_*$ -comodules:

$$\begin{array}{ccccc} M_2(j) \otimes \mathbb{F}_2\{1\} & \xrightarrow{\alpha} & N_1(2j) & \xrightarrow{\beta_1} & K \\ \downarrow & & \downarrow \delta & & \downarrow \beta_2 \\ M_2(j) \otimes (A(2)//A(1))_* & \longrightarrow & Q^j(A//A(1))_* & \longrightarrow & Q^{j-1}(A//A(1))_* \end{array}$$

We deduce that the map  $\beta$  is a map of  $A(2)_*$ -comodules, because it is given by the composite  $\beta_2 \circ \beta_1$  of  $A(2)_*$ -comodule maps. □

**Vanishing lines.** We reduce the computations needed to those of  $M_2(1)^{\otimes k}$  for  $k \leq 3$  using vanishing lines for modified Adams  $E_2$  terms. Note that after a finite range,  $\text{Ext}_{A(2)_*}(H(1, 4))$  has a vanishing line of slope  $1/5$ .

**Lemma 7.3.** We have

$$\text{Ext}_{A(2)_*}^{s,t}(N_1(j) \otimes H(1, 4)) = 0$$

for

$$s > \max \left\{ \frac{(t-s)+17}{7}, \frac{(t-s)+a_j}{6}, \frac{(t-s)+b_j}{5} \right\}$$

and the constants  $a_j$  and  $b_j$  are inductively defined by

$$\begin{aligned} a_0 &= 21, \\ b_0 &= 9, \\ a_1 &= 15, \\ b_1 &= 2, \\ a_{2j} &= \max\{a_{j-1} - 8j - 2, a_j - 8j\}, \\ b_{2j} &= \max\{b_{j-1} - 8j - 3, b_j - 8j\}, \\ a_{2j+1} &= a_j - 8j, \\ b_{2j+1} &= b_j - 8j. \end{aligned}$$

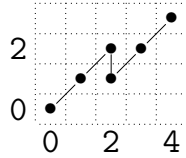
*Proof.* The case of  $j = 0, 1$  is obtained by examining Figures 6.4 and 8.1. The case of  $j \geq 2$  is established by induction using Lemmas 7.1 and 7.2. The terms involving  $Q^j(A//A(1))_*$  are handled using the spectral sequence

$$\text{Ext}_{A(2)_*}^{s,t}(N_2(j) \otimes (A(2)//A(1))_* \otimes H(1, 4)) \Rightarrow \text{Ext}_{A(2)_*}^{s,t}(Q^j(A//A(1))_* \otimes H(1, 4))$$

induced from (7.1), and the change-of-rings isomorphism

$$\text{Ext}_{A(2)_*}^{s,t}(N_2(j) \otimes (A(2)//A(1))_* \otimes H(1, 4)) \cong \text{Ext}_{A(1)_*}^{s,t}(N_2(j) \otimes H(1, 4)).$$

The only non-zero values of  $\text{Ext}_{A(1)_*}^{s,t}(H(1, 4))$  are displayed below.



In particular, we see that  $\text{Ext}_{A(1)_*}^{s,t}(H(1, 4))$  is zero for  $s > \frac{(t-s)+17}{7}$ .  $\square$

We extract the following estimate.

**Lemma 7.4.** Suppose that  $j_1, \dots, j_n$  is a sequence of positive integers such that for some  $i$ ,  $j_i \geq 2$ . Then we have

$$\text{Ext}_{A(2)_*}^{s,t}(M_2(j_1) \otimes \dots \otimes M_2(j_n)[-n] \otimes H(1, 4)) = 0$$

for

$$s > \max \left\{ \frac{(t-s)+17}{7}, \frac{(t-s)+2}{6}, \frac{(t-s)-12}{5} \right\}.$$

*Proof.* Assume that  $n = 1$ , and set  $j$  equal to  $j_1 \geq 2$ . By Lemma 7.3, we have

$$\mathrm{Ext}_{A(2)_*}(M_2(j)[-1] \otimes H(1, 4)) = 0$$

if

$$s > \max \left\{ \frac{(t-s) - 8j + 8 + 17}{7}, \frac{(t-s) - 8j + 7 + a_j}{6}, \frac{(t-s) - 8j + 6 + b_j}{5} \right\}.$$

It therefore suffices to prove that the following inequalities are satisfied:

$$\begin{aligned} 17 &\geq 17 - 8j + 8, \\ 2 &\geq a_j - 8j + 7, \\ -12 &\geq b_j - 8j + 6. \end{aligned}$$

The inequalities are true for  $j = 2, 3$ . By induction, these inequalities hold for all  $j$ .

We now induct on  $n$ . We may as well assume that  $j_1 \geq 2$ . Assume that

$$\mathrm{Ext}_{A(2)_*}^{s,t}(M_2(j_1) \otimes \cdots \otimes M_2(j_{n-1})[-n+1] \otimes H(1, 4)) = 0$$

for

$$s > \max \left\{ \frac{(t-s) + 17}{7}, \frac{(t-s) + 2}{6}, \frac{(t-s) - 12}{5} \right\}.$$

By filtering the  $A_*$ -comodule  $M_2(j_n)$  by degree, we obtain an Atiyah-Hirzebruch type spectral sequence which converges to

$$\mathrm{Ext}_{A(2)_*}^{s,t}(M_2(j_1) \otimes \cdots \otimes M_2(j_n)[-n] \otimes H(1, 4))$$

and whose  $E_1$ -page is given by

$$\bigoplus_x \mathrm{Ext}_{A(2)_*}^{s,t}(\Sigma^{|x|} M_2(j_1) \otimes \cdots \otimes M_2(j_{n-1})[-n] \otimes H(1, 4)).$$

where  $x$  ranges over an  $\mathbb{F}_2$ -basis of  $M_2(j_n)$ . The smallest value  $|x|$  can take is 8, in the case  $j_1 = 1$ . By our inductive hypothesis, we have

$$\mathrm{Ext}_{A(2)_*}^{s,t}(\Sigma^8 M_2(j_1) \otimes \cdots \otimes M_2(j_{n-1})[-n] \otimes H(1, 4)) = 0$$

for

$$s > \max \left\{ \frac{(t-s) + 17}{7}, \frac{(t-s) + 1}{6}, \frac{(t-s) - 14}{5} \right\}.$$

This verifies the inductive step.  $\square$

**Lemma 7.5.** Suppose that  $n$  is greater than 3. Then we have

$$\mathrm{Ext}_{A(2)_*}^{s,t}(M_2(1)^{\otimes n}[-n] \otimes H(1, 4)) = 0$$

for

$$s > \max \left\{ \frac{(t-s) + 17}{7}, \frac{(t-s) + 4}{6}, \frac{(t-s) - 17}{5} \right\}.$$

*Proof.* Examining Figure 6.5, we see that

$$\mathrm{Ext}_{A(2)_*}^{s,t}(N_1(1)^{\otimes 3} \otimes H(1, 4)) = 0$$

for

$$s > \max \left\{ \frac{(t-s) + 17}{7}, \frac{(t-s) + 8}{6}, \frac{(t-s) - 9}{5} \right\}.$$

The lemma follows from induction on  $n$ , using Atiyah-Hirzebruch type spectral sequences as in the proof of Lemma 7.5.  $\square$

8. THE MODIFIED ADAMS SPECTRAL SEQUENCE FOR  $tmf_*M(1, 4)$ 

In this section we describe a complete computation of the MASS

$$\mathrm{Ext}_{A(2)_*}^{s,t}(H(1, 4)) \Rightarrow \pi_{t-s}(tmf \wedge M(1, 4)).$$

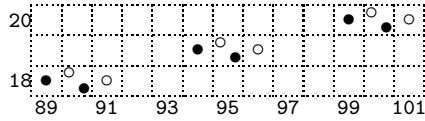
The spectral sequence is displayed in four pages in Figures 8.1 and 8.2. The entire spectral sequence is  $v_2^{32}$ -periodic.

We first explain what is happening in these charts. Then we explain the methodology used to produce these differentials.

**Page 1: dimensions 0–48.** The truncated wedge beginning in  $t - s = 35$  is infinite, and propagated by  $g$ -multiplication. The entire chart is periodic under  $v_2^8$ -multiplication. Classes born on the 0-cell of  $M(1)$  are denoted with a  $\bullet$ , and classes born on the 1-cell of  $M(1)$  are denoted with a  $\circ$ . Although multiplication by  $c_4 = v_1^4$  is faithful in  $\mathrm{Ext}_{A(2)_*}(\mathbb{F}_2)$ , it is not faithful in  $\mathrm{Ext}_{A(2)_*}(M(1))$ . We therefore get some classes coming from the 9-cell of  $M(1, 4)$ , which we denote with a  $\diamond$ .

There are only two possible Adams differentials through  $t - s = 47$ , and only one of them actually occurs. This differential is indicated on the chart.

**Page 2: dimensions 48–96.** We move to the region between the occurrence of  $v_2^8$  and  $v_2^{16}$ . There are numerous  $d_2$  differentials in this range, displayed in the chart. In this chart, the classes propagated by  $v_2^8$  are denoted with a  $\bullet$ , and the classes coming from the truncated wedge starting in  $t - s = 35$  are denoted with a  $\circ$ . Note that beginning in  $t - s = 91$ , we just have the following pattern.



**Page 3: dimensions 96–144.** We now move up to the region between  $v_2^{16}$  and  $v_2^{24}$ . We propagate only the  $h_{2,1}$ -periodic pattern from the previous page (denoted with  $\circ$ ); everything else is either the source or target of a  $d_2$ . We denote the elements propagated by  $v_2^{16}$  multiplication with a  $\bullet$ .

**Page 4: dimensions 144–192.** We now introduce the differentials supported by  $v_2^{24}$  and its multiples. We see that eventually we get a small gap in homotopy between the 180 stem and the 192 stem. Then the pattern repeats with  $v_2^{32}$ -periodicity.

**Methodology.** In [HM], the structure of the Adams spectral sequence

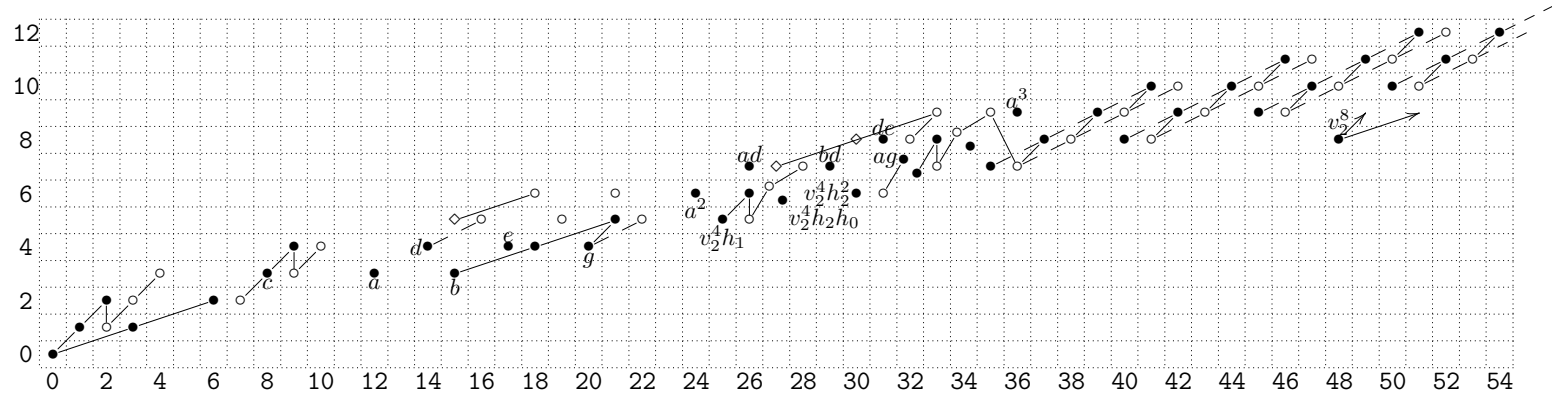
$$\mathrm{Ext}_{A(2)_*}^{*,*}(\mathbb{F}_2) \Rightarrow \pi_* tmf_2$$

is completely determined. The Adams spectral sequence for  $tmf_*M(1, 4)$  is a module over the Adams spectral sequence for  $tmf_*$ , and all of the differentials for  $tmf_*M(1, 4)$  were deduced from this structure. These computations were double-checked against the Atiyah-Hirzebruch spectral sequence

$$H^*(M(1, 4), tmf_*) \Rightarrow tmf_*M(1, 4)$$

using the known values of  $tmf_*$ . As a further consistency check, a combination of Gross-Hopkins duality [HG94] and Mahowald-Rezk [MR99] duality shows that

MASS for  $tmf_*M(1, 4)$ , p1:



MASS for  $tmf_*M(1, 4)$ , p2:

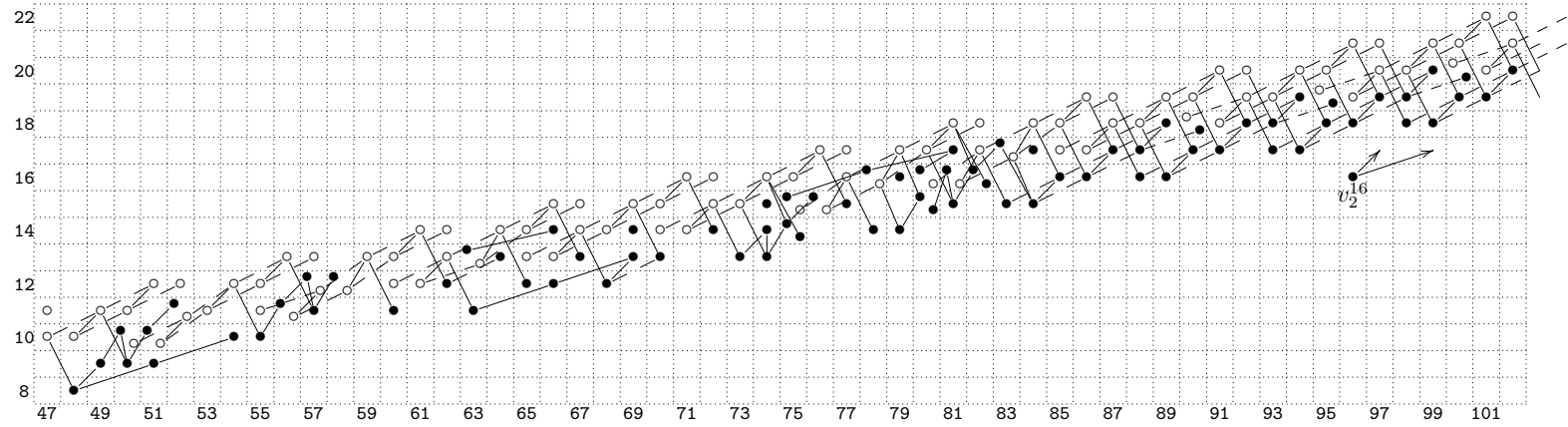
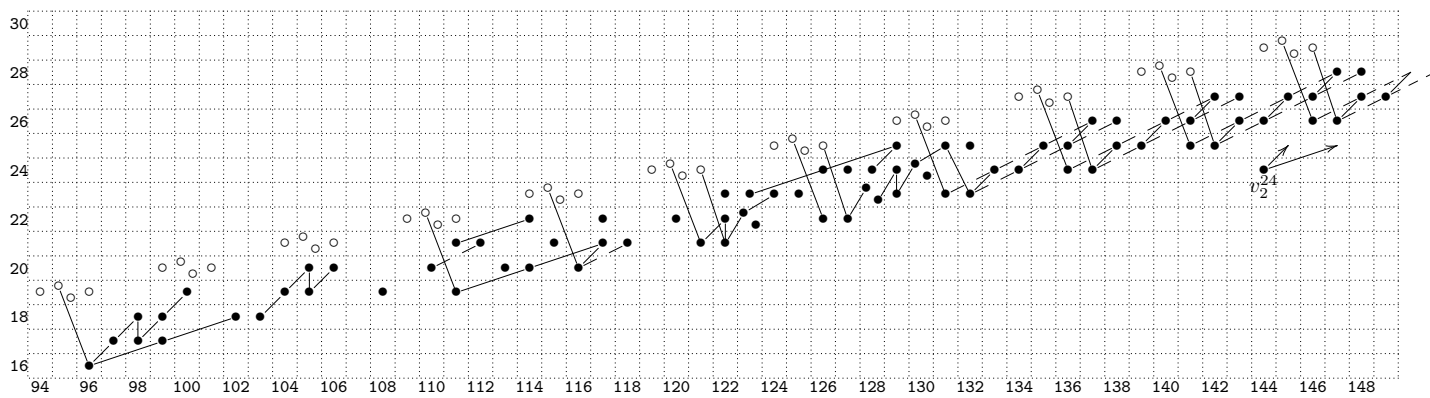


FIGURE 8.1.



MASS for  $tmf_*M(1,4)$ , p3:



MASS for  $tmf_*M(1,4)$ , p4:

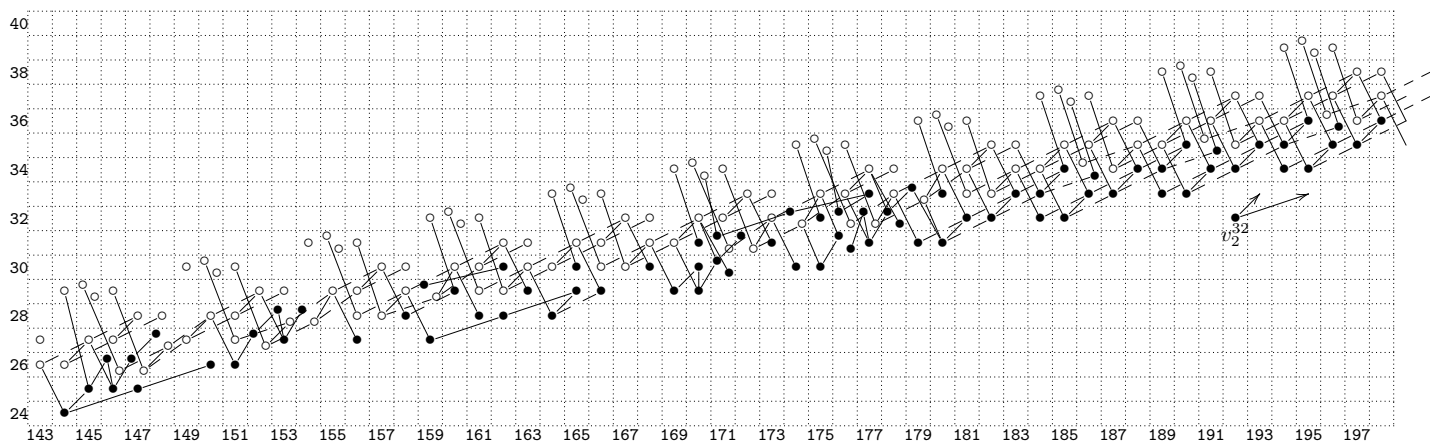


FIGURE 8.2.

$tmf_*M(1, 4)$  is, up to a shift, Pontryagin self-dual, and this is consistent with our computations.

### 9. $d_2(v_2^8)$ AND $d_3(v_2^{16})$

In this section we will lift the differentials  $d_2(v_2^8)$  and  $d_3(v_2^{16})$  from the MASS for  $tmf_*M(1, 4)$  to the MASS for  $\pi_*(M(1, 4) \wedge DM(1, 4))$ . We will observe that both  $d_2(v_2^8)$  and  $d_3(v_2^{16})$  are central, and hence, using the fact that the MASS for  $\pi_*(M(1, 4) \wedge DM(1, 4))$  is a spectral sequence of algebras, we will deduce that  $d_r(v_2^{32})$  is zero for  $r < 4$ .

**Lemma 9.1.** In the MASS for  $\pi_*(M(1, 4) \wedge DM(1, 4))$ , there is a differential

$$d_2(v_2^8) = \widetilde{e_0}r,$$

where  $\widetilde{e_0}r$  is the image of the element  $e_0r$  under the map

$$\text{Ext}_{A_*}^{10,57}(\mathbb{F}_2) \rightarrow \text{Ext}_{A_*}^{10,57}(H(1, 4) \otimes DH(1, 4)).$$

*Proof.* By Proposition 2.1 it suffices to establish that  $d_2(v_2^8) = \widetilde{e_0}r$  in the MASS

$$\text{Ext}_{A_*}^{s,t}(H(1, 4) \otimes DH(1)) \Rightarrow \pi_{t-s}(M(1, 4) \wedge DM(1)).$$

The differential  $d_2(v_2^8)$  in the Adams spectral sequence for  $tmf$  maps to a differential  $d_2(v_2^8) = \widetilde{e_0}r$  under the map of (M)ASSs

$$\begin{array}{ccc} \text{Ext}_{A(2)_*}^{s,t}(\mathbb{F}_2) & \xRightarrow{\quad\quad\quad} & \pi_{t-s}tmf \\ \downarrow & & \downarrow \\ \text{Ext}_{A(2)_*}^{s,t}(H(1, 4) \otimes DH(1)) & \xRightarrow{\quad\quad\quad} & tmf_{t-s}(M(1, 4) \wedge DM(1)) \end{array}$$

where  $\widetilde{e_0}r$  is the image of  $e_0r$  under the composite

$$\text{Ext}_{A_*}^{10,57}(\mathbb{F}_2) \rightarrow \text{Ext}_{A_*}^{10,57}(H(1, 4) \otimes DH(1)) \rightarrow \text{Ext}_{A(2)_*}^{10,57}(H(1, 4) \otimes DH(1)).$$

We wish to lift the differential  $d_2(v_2^8) = \widetilde{e_0}r$  to  $d_2(v_2^8) = \widetilde{e_0}r$  using the map of MASSs:

$$\begin{array}{ccc} \text{Ext}_{A_*}^{s,t}(H(1, 4) \otimes DH(1)) & \xRightarrow{\quad\quad\quad} & \pi_{t-s}(M(1, 4) \wedge DM(1)) \\ \downarrow & & \downarrow \\ \text{Ext}_{A(2)_*}^{s,t}(H(1, 4) \otimes DH(1)) & \xRightarrow{\quad\quad\quad} & tmf_{t-s}(M(1, 4) \wedge DM(1)) \end{array}$$

However, using

$$\text{Ext}_{A(2)_*}^{s,t}(H(1, 4)) = \begin{cases} e_0r[0] & (t-s, s) = (47, 10), \\ e_0r[1] & (t-s, s) = (48, 10) \end{cases}$$

and

$$\text{Ext}_{A_*}^{s,t}(\mathbb{F}_2) = \begin{cases} e_0r & (t-s, s) = (47, 10), \\ 0 & (t-s, s) = (46, 10), (48, 10), (55, 5), (56, 5), (57, 5) \end{cases}$$

we may deduce that the map

$$\text{Ext}_{A_*}^{10,57}(H(1, 4) \otimes DH(1)) \rightarrow \text{Ext}_{A(2)_*}^{10,57}(H(1, 4) \otimes DH(1))$$

is an isomorphism. This suffices to show that the differential  $d_2(v_2^8)$  lifts as desired.  $\square$

Since  $d_2(v_2^8)$  is central, Proposition 3.2 gives the following corollary.

**Corollary 9.2.** In the MASS for  $\pi_*(M(1, 4) \wedge DM(1, 4))$ , we have  $d_2(v_2^{16}) = 0$ .

We now investigate  $d_3(v_2^{16})$ .

**Lemma 9.3.** In the MASS for  $\pi_*(M(1, 4) \wedge DM(1, 4))$ , the element  $d_3(v_2^{16})$  is in the image of

$$\mathrm{Ext}_{A_*}^{19,114}(\mathbb{F}_2) \rightarrow \mathrm{Ext}_{A_*}^{19,114}(H(1, 4) \otimes DH(1, 4)).$$

In particular,  $d_3(v_2^{16})$  is central.

*Proof.* By Proposition 2.1 it suffices to establish that in the MASS for  $\pi_*(M(1, 4) \wedge DM(1))$ , the element  $y = d_3(v_2^{16})$  is in the image of the map

$$\mathrm{Ext}_{A_*}^{19,114}(\mathbb{F}_2) \rightarrow \mathrm{Ext}_{A_*}^{19,114}(H(1, 4) \otimes DH(1)).$$

The differential  $d_3(v_2^{16})$  in the ASS for  $tmf$  maps to a differential  $d_3(v_2^{16}) = z$  under the map of (M)ASSs

$$\begin{array}{ccc} \mathrm{Ext}_{A(2)_*}^{s,t}(\mathbb{F}_2) & \xRightarrow{\quad} & \pi_{t-s}tmf \\ \downarrow & & \downarrow \\ \mathrm{Ext}_{A(2)_*}^{s,t}(H(1, 4) \otimes DH(1)) & \xRightarrow{\quad} & tmf_{t-s}(M(1, 4) \wedge DM(1)) \end{array}$$

where  $z$  is in the image of

$$\mathrm{Ext}_{A(2)_*}^{19,114}(\mathbb{F}_2) \rightarrow \mathrm{Ext}_{A(2)_*}^{19,114}(H(1, 4) \otimes DH(1)).$$

Using the map of spectral sequences

$$\begin{array}{ccc} \mathrm{Ext}_{A_*}^{s,t}(H(1, 4) \otimes DH(1)) & \xRightarrow{\quad} & \pi_{t-s}(M(1, 4) \wedge DM(1)) \\ \downarrow & & \downarrow \\ \mathrm{Ext}_{A(2)_*}^{s,t}(H(1, 4) \otimes DH(1)) & \xRightarrow{\quad} & tmf_{t-s}(M(1, 4) \wedge DM(1)) \end{array}$$

we see that  $y$  maps to  $z$ . Therefore  $z$  detects  $y$  in the algebraic  $tmf$ -resolution for  $\mathrm{Ext}_{A_*}^{*,*}(H(1, 4) \otimes DH(1))$ . Since the algebraic  $tmf$ -resolution is functorial, we deduce that  $y$  is in the image of the map

$$\mathrm{Ext}_{A_*}^{19,114}(\mathbb{F}_2) \xrightarrow{i_*} \mathrm{Ext}_{A_*}^{19,114}(H(1, 4) \otimes DH(1))$$

modulo higher terms of the algebraic  $tmf$ -resolution: that is to say, there exists an element

$$x \in \mathrm{Ext}_{A_*}^{19,114}(\mathbb{F}_2)$$

such that  $y - i_*(x)$  is detected in a higher filtration of the algebraic  $tmf$ -resolution.

We are left with showing that  $w = y - i_*(x) = 0$ . Suppose not. Using our vanishing lines from Section 7 and our  $\mathrm{Ext}_{A(2)_*}$  computations from Section 6, we deduce that  $w$  is detected in the algebraic  $tmf$ -resolution by an element

$$\overline{w} \in \mathrm{Ext}_{A(2)_*}^{19,114}(M_2(1) \otimes H(1, 4) \otimes DH(1)[-1])$$

and the image of  $\overline{w}$  under the map

$$\mathrm{Ext}_{A(2)_*}^{19,114}(M_2(1) \otimes H(1,4) \otimes DH(1)[-1]) \xrightarrow{1 \otimes p_* \otimes 1} \mathrm{Ext}_{A(2)_*}^{19,114}(M_2(1) \otimes \Sigma^{12} H(1) \otimes DH(1)[-4])$$

is non-trivial, where  $p_*$  is the projection

$$H(1,4) \rightarrow \Sigma^{12} H(1)[-3]$$

in the derived category of  $A_*$ -comodules induced by the projection

$$p : M(1,4) \rightarrow \Sigma^9 M(1).$$

We deduce that in the MASS for  $M(1) \wedge DM(1)$  there is a differential

$$d_3((p_* \otimes 1)(v_2^{16})) = (p_* \otimes 1)(w).$$

We will verify the following claim:

**Claim 9.4.** The element  $(p_* \otimes 1)(w)$  is non-trivial in the  $E_3$ -page of the MASS for  $M(1) \wedge DM(1)$ .

Assuming Claim 9.4, we deduce that  $d_3((p_* \otimes 1)(v_2^{16}))$  is non-trivial. However, the image of  $v_2^{16}$  under the map

$$\mathrm{Ext}_{A_*}^{16,112}(H(1,4) \otimes DH(1)) \xrightarrow{p_* \otimes 1} \mathrm{Ext}_{A_*}^{16,112}(\Sigma^{12} H(1) \otimes DH(1)[-3])$$

may be computed using the May spectral sequence. In the May spectral sequence, the element  $v_2^{16}$  is detected by  $b_{3,0}^8$ . Applying Nakamura's formula [Nak72] to the May spectral sequence differential  $d_8(b_{3,0}^4) = h_5 b_{2,0}^4$  in the proof of Proposition 4.3 gives

$$d_{16}(b_{3,0}^8) = h_6 b_{2,0}^8$$

from which it follows that

$$(p_* \otimes 1)(v_2^{16}) = h_6 b_{2,0}^6.$$

The element  $b_{2,0}^6$  detects the cube of the Adams map:

$$v_1^{12} = (v_1^4)^3 \in \pi_{24}(M(1) \wedge DM(1)).$$

Since this homotopy element has order 2, the Adams differential

$$d_2(h_6) = h_0 h_5^2$$

implies that the element  $h_6 b_{2,0}^6$  detects the Toda bracket of the composite

$$S^{86} \xrightarrow{\theta_5} S^{24} \xrightarrow{2} S^{24} \xrightarrow{v_1^{12}} M(1) \wedge DM(1).$$

In particular,  $h_6 b_{2,0}^6$  is a permanent cycle in the MASS for  $M(1) \wedge DM(1)$ , which contradicts the existence of a non-trivial differential  $d_3((p_* \otimes 1)(v_2^{16}))$ . Thus the assumption that  $w \neq 0$  gives rise to a contradiction, and we conclude that  $w = 0$ , as desired.

We are left with verifying Claim 9.4. We will verify this claim by establishing:

(1) The element

$$(1 \otimes p_* \otimes 1)(\overline{w}) \in \mathrm{Ext}_{A(2)_*}^{19,114}(M_2(1) \otimes \Sigma^{12} H(1) \otimes DH(1)[-4])$$

is not the target of a differential in the algebraic  $tmf$ -resolution for  $\mathrm{Ext}_{A_*}^{*,*}(H(1) \otimes DH(1))$ .

(2) The element

$$(p_* \otimes 1)(w) \in \text{Ext}_{A_*}^{19,114}(\Sigma^{12} H(1) \otimes DH(1)[-3])$$

is not the target of a  $d_2$  differential in the MASS for  $M(1) \wedge DM(1)$ .

Item (1) above is verified by observing that

$$\text{Ext}_{A(2)_*}^{s,t}(\mathbb{F}_2) = 0 \quad (t-s, s) = (86, 15), (87, 15), (88, 15)$$

and so there are no possible contributions to

$$\text{Ext}_{A(2)_*}^{87,15}(H(1) \otimes DH(1)),$$

and this is the only possible source for a differential in the algebraic  $tmf$ -resolution.

We now verify (2). The  $A_*$ -comodule  $H(1) \otimes DH(1)$  has the following diagram of generators.

$$(9.1) \quad \begin{array}{ccc} 1 & \circ & \\ 0 & \bullet & \triangle \\ -1 & & \square \end{array} \quad \begin{array}{l} \downarrow \\ \downarrow \\ \downarrow \end{array} \quad \begin{array}{l} \downarrow \\ \downarrow \\ \downarrow \end{array} \quad \begin{array}{l} \downarrow \\ \downarrow \\ \downarrow \end{array}$$

Here the straight lines encode the action of  $Sq_*^1$  and the curved line denotes a  $Sq_*^2$ . Using Bruner's computer generated  $\text{Ext}_{A_*}(\mathbb{F}_2)$  charts [Bru93], we compute the vicinity of  $(p_* \otimes 1)(w)$  in  $\text{Ext}_{A_*}^{*,*}(H(1) \otimes DH(1))$  in Table 9.1.

| $s \setminus t-s$ | 86   | 87                                 |
|-------------------|--|------------------------------------|
| 16                | $\circ \circ$<br>$\triangle$<br>$(p_* \otimes 1)(w) \bullet \bullet$ | *                                  |
| 15                | *  | *                                  |
| 14                | *  | $\circ b_{87}$<br>$\square a_{87}$ |

TABLE 9.1.  $\text{Ext}_{A_*}^{s,t}(H(1) \otimes DH(1))$  near  $(p_* \otimes 1)(w)$

In this table, entries marked with \* are not computed, otherwise, elements are denoted by the generator (as in (9.1)) that supports it. The only possible sources for a non-trivial  $d_2$  are  $a_{87}$  and  $b_{87}$ .

The element  $a_{87}$  is the image of an element

$$a_{88} \in \text{Ext}_{A_*}^{14,102}(\mathbb{F}_2)$$

under the inclusion of the bottom generator

$$\Sigma^{-1}\mathbb{F}_2 \rightarrow H(1) \otimes DH(1).$$

Since  $\text{Ext}_{A_*}^{16,103}(\mathbb{F}_2) = 0$ , we deduce that  $d_2(a_{88}) = 0$  in the ASS for  $\pi_* S$ . The map of MASSs induced from the inclusion of the bottom cell of  $M(1) \wedge DM(1)$  gives  $d_2(a_{87}) = 0$ .

We now turn our attention to  $b_{87}$ . Table 9.2 shows the portion of  $\text{Ext}_{A_*}(H(1) \otimes DH(1))$  mapped to the vicinity of Table 9.1 under  $h_2$ -multiplication.

| $s \setminus t - s$ | 83   | 84                          |
|---------------------|--|-----------------------------|
| 15                  | $c_{83} \bullet$<br>$c'_{83} \square \square c''_{83}$ | *                           |
| 14                  | *  | *                           |
| 13                  | *  | $\circ b_{84}$<br>$\square$ |

TABLE 9.2.  $\text{Ext}_{A_*}^{s,t}(H(1) \otimes DH(1))$  near  $h_2^{-1}b_{87}$ 

Using the  $h_2$  multiplicative structure in Bruner's tables [Bru93], we deduce that

$$\begin{aligned} h_2 b_{84} &= b_{87}, \\ h_2 c_{83} &= 0, \\ h_2 c'_{83} &= 0, \\ h_2 c''_{83} &= 0. \end{aligned}$$

Since  $h_2$  is a permanent cycle in the ASS for the sphere, we have

$$d_2(b_{87}) = d_2(h_2 b_{84}) = h_2 d_2(b_{84}) = 0.$$

This completes our proof of Claim 9.4.  $\square$

Proposition 3.2 gives the following corollary.

**Corollary 9.5.** In the MASS for  $\pi_*(M(1,4) \wedge DM(1,4))$ , we have  $d_3(v_2^{32}) = 0$ .

## 10. CALCULATION OF AN ADAMS DIFFERENTIAL

The image of the element  $\bar{\kappa} \in \pi_{20}(S)_2$  in  $\pi_{20}(M(1,4) \wedge DM(1,4))$  gives rise to a self-map

$$\tilde{\kappa} : M(1,4) \rightarrow M(1,4).$$

The element  $g \in \text{Ext}_{A_*}^{4,24}(\mathbb{F}_2)$  which detects  $\bar{\kappa}$  maps to a permanent cycle  $\tilde{g} \in \text{Ext}_{A_*}(H(1,4) \otimes DH(1,4))$  which detects  $\tilde{\kappa} \in \pi_{20}(M(1,4) \wedge DM(1,4))$  in the MASS. The purpose of this section is to prove the following theorem.

**Theorem 10.1.**

- (1) The element  $v_2^{20}h_1 \in \text{Ext}_{A(2)_*}^{21,142}(H(1,4))$  lifts to an element

$$\widetilde{v_2^{20}h_1} \in \text{Ext}_{A_*}^{21,142}(H(1,4) \otimes DH(1,4)).$$

- (2) There is a differential

$$d_3(\widetilde{v_2^{20}h_1}) = \tilde{g}^6 + R$$

in the MASS for  $M(1,4) \wedge DM(1,4)$ , where  $R$  is an element of filtration greater than 0 in the algebraic  $tmf$ -resolution.

*Proof.* Table 10.1 displays a small portion of the  $E_1$ -page of the algebraic  $tmf$ -resolution for  $\text{Ext}_{A_*}(H(1,4) \wedge DH(1))$ .

| $s \setminus t - s$ | 120   | 121   |
|---------------------|---|---|
| 24                  | $\bullet \bullet \bullet g^6$<br>$\circ \circ$  | $\bullet \bullet$<br>$\circ$  |
| 23                  | $a_{120} \bullet \bullet b_{120}$<br>$\circ x_{120}$<br>$\odot y_{120}$                     | $\bullet \bullet$<br>$\circ \circ$<br>$\odot \odot$                                 |
| 22                  | $\bullet$<br>$\odot z_{120}$  | $\odot$<br>$*$  |
| 21                  | $\bullet$<br>$\circ$<br>$\odot \odot \odot \odot \odot$<br>$\odot \odot \odot \odot$<br>$*$ | $v_2^{20} h_1 \bullet \bullet$<br>$\odot \odot \odot \odot$<br>$\odot \odot$<br>$*$ |

TABLE 10.1. The algebraic  $tmf$ -resolution for  $\text{Ext}_{A_*}(H(1, 4) \otimes DH(1))$  near  $v_2^{20} h_1$ 

In this and all future tables depicting the algebraic  $tmf$ -resolution, we have the following key:

- $\bullet$  = generator of  $\text{Ext}_{A(2)_*}(H(1, 4) \otimes DH(1))$ ,
- $\circ$  = generator of  $\text{Ext}_{A(2)_*}(M_2(1)[-1] \otimes H(1, 4) \otimes DH(1))$ ,
- $\odot$  = generator of  $\text{Ext}_{A(2)_*}(M_2(1)^{\otimes 2}[-2] \otimes H(1, 4) \otimes DH(1))$ ,
- $\odot$  = generator of  $\text{Ext}_{A(2)_*}(M_2(1)^{\otimes 3}[-3] \otimes H(1, 4) \otimes DH(1))$ ,
- $*$  = potential contribution from

$$\text{Ext}_{A(2)_*}(M_2(j_1) \otimes \cdots \otimes M_2(j_n)[-n] \otimes H(1, 4) \otimes DH(1))$$

where either for some  $i$ ,  $j_i > 1$ , or  $n > 3$ .

We shall refer to all differentials in the algebraic  $tmf$ -resolution as  $d_1$  differentials. Differentials in the MASS

$$E_2^{s,t} = \text{Ext}_{A_*}^{s,t}(H(1, 4) \otimes DH(1)) \Rightarrow \pi_{t-s}(M(1, 4) \wedge DM(1))$$

will be referred to by  $d_r$  for  $r \geq 2$ .

In order to prove (1), we must show that the element  $v_2^{20} h_1$  in Table 10.1 does not support a non-trivial  $d_1$ . There is one possible target  $z_{120}$  in  $(t - s, s) = (120, 22)$ , but we will argue shortly that this possibility cannot occur. Assuming for the moment that  $d_1(v_2^{20} h_1) = 0$ , we would conclude that  $v_2^{20} h_1$  lifts to an element

$$\widetilde{v_2^{20} h_1} \in \text{Ext}_{A_*}^{21,142}(H(1, 4) \otimes DH(1, 4)).$$

The composite

$$H(1, 4) \wedge DH(1, 4) \rightarrow H(1, 4) \rightarrow tmf \wedge H(1, 4)$$

induces a map of MASSs:

$$\begin{array}{ccc} \mathrm{Ext}_{A_*}^{s,t}(H(1,4) \otimes DH(1,4)) & \Longrightarrow & \pi_{t-s}(M(1,4) \wedge DM(1,4)) \\ \downarrow & & \downarrow \\ \mathrm{Ext}_{A(2)_*}^{s,t}(H(1,4)) & \Longrightarrow & \pi_{t-s}(tmf \wedge M(1,4)) \end{array}$$

In the MASS for  $tmf \wedge M(1,4)$ , there is a differential

$$d_3(v_2^{20}h_1) = g^6.$$

In order to prove (2), we need to lift this differential to the MASS for  $M(1,4) \wedge DM(1,4)$ . By Proposition 2.1, it suffices to lift this differential to the MASS for  $M(1,4) \wedge DM(1)$ :

$$\mathrm{Ext}_{A_*}^{s,t}(H(1,4) \otimes DH(1)) \Rightarrow \pi_{t-s}(M(1,4) \wedge DM(1)).$$

The obstruction to lifting this differential is that  $\widetilde{v_2^{20}h_1}$  could support a  $d_2$  in the MASS for  $M(1,4) \wedge DM(1)$ . In fact, Table 10.1 demonstrates that there are four possible targets for such a  $d_2$  in  $(t-s, s) = (120, 23)$ : these are labeled  $a_{120}$ ,  $b_{120}$ ,  $x_{120}$ ,  $y_{120}$ .

We now argue (1) and (2) by showing that the element  $v_2^{20}h_1$  in Table 10.1 cannot support a non-trivial  $d_1$  or  $d_2$ . We will need Tables 10.2 and 10.3, which depict the  $tmf$ -resolution in the vicinities of  $gv_2^{20}h_1$  and  $gv_2^4h_1$ , respectively.

| $s \setminus t-s$ | 140                                 | 141                            |
|-------------------|-------------------------------------|--------------------------------|
| 27                | $ga_{120} \bullet \bullet gb_{120}$ | $\bullet$                      |
|                   | $\circ gx_{120}$                    | $\circ \circ$                  |
|                   | $\odot gy_{120}$                    | $\odot \odot$                  |
|                   |                                     |                                |
| 26                | $\bullet \bullet$                   | $\bullet x_{141}$              |
|                   | $\odot gz_{120}$                    | $\odot$                        |
|                   |                                     | $*$                            |
| 25                | $\bullet$                           | $gv_2^{20}h_1 \bullet \bullet$ |
|                   | $\circ \circ \circ$                 | $\circ \circ$                  |
|                   | $\odot \odot \odot$                 | $\odot \odot$                  |
|                   | $\odot \odot \odot \odot$           | $\odot \odot$                  |
|                   | $*$                                 | $*$                            |
|                   |                                     |                                |

TABLE 10.2. The algebraic  $tmf$ -resolution for  $\mathrm{Ext}_{A_*}(H(1,4) \otimes DH(1))$  near  $gv_2^{20}h_1$

Write  $d_1(v_2^{20}h_1) = c \cdot z_{120}$  for  $c \in \mathbb{F}_2$ . Then have

$$d_1(gv_2^{20}h_1) = c \cdot gz_{120}.$$

Table 10.3 shows that  $d_1(gv_2^4h_1) = 0$ . Multiplying by the  $d_2$ -cycle  $v_2^{16}$  of Corollary 9.2, we deduce that we must have  $d_1(gv_2^{20}h_1) = 0$ . Thus  $c$  equals 0, and we have proven (1).

Write

$$d_2(v_2^{20}h_1) = c_1a_{120} + c_2b_{120} + c_3x_{120} + c_4y_{120}$$



| $s \setminus t - s$ | 44      | 45                         |
|---------------------|---------|----------------------------|
| 11                  |         | •                          |
| 10                  | ••      | • $v_2^{-16} x_{141}$      |
| 9                   | •<br>○○ | $gv_2^4 h_1$ ••<br>○○<br>* |

TABLE 10.3. The algebraic  $tmf$ -resolution for  $\text{Ext}_{A_*}(H(1, 4) \otimes DH(1))$  near  $gv_2^4 h_1$ 

for  $c_i \in \mathbb{F}_2$ . The image of  $v_2^{20} h_1$  in  $\text{Ext}_{A(2)_*}(H(1, 4))$  is a  $d_2$ -cycle in the MASS

$$\text{Ext}_{A(2)_*}^{s,t}(H(1, 4)) \rightarrow \pi_{t-s}(tmf \wedge M(1, 4)).$$

We therefore deduce that  $c_1 = c_2 = 0$ . We wish to show that  $d_2(v_2^{20} h_1) = 0$ , i.e. that it is contained in the image of  $d_1$ . We have

$$d_2(gv_2^{20} h_1) = c_3 g x_{120} + c_4 g y_{120}.$$

Examining Table 10.3, we see that  $d_2(gv_2^4 h_1) = 0$ . Since  $v_2^{16}$  is a  $d_2$ -cycle, we deduce that  $d_2(gv_2^{20} h_1) = 0$ . This means that

$$c_3 g x_{120} + c_4 g y_{120}$$

is in the target of a  $d_1$ . With the exception of the element  $x_{141}$ , all of the generators in  $(t - s, s) = (141, 26)$  are  $g$ -periodic. Thus we have

$$d_1(E_1^{26,167}) = g \cdot d_1(E_1^{22,143}) + \mathbb{F}_2\{d_1(x_{141})\}$$

where  $E_1^{s,t}$  is the  $E_1$ -term of the algebraic  $tmf$ -resolution for  $\text{Ext}_{A_*}^{*,*}(H(1, 4) \otimes DH(1))$ . However, we see from Table 10.3 that  $d_1(v_2^{-16} x_{141}) = 0$ , so it follows that  $d_1(x_{141}) = 0$ . We may therefore deduce the vanishing of  $d_2(v_2^{20} h_1)$  from the vanishing of  $d_2(gv_2^{20} h_1)$ . We have proven (2).  $\square$

## 11. PROOF OF THE MAIN THEOREM

By Proposition 4.3 and Lemma 5.7, the element

$$v_2^{32} \in \text{Ext}_{A(2)_*}^{32,224}(H(1, 4) \otimes DH(1))$$

is a permanent cycle in the algebraic  $tmf$ -resolution, and it detects an element

$$v_2^{32} \in \text{Ext}_{A_*}^{32,224}(H(1, 4) \otimes DH(1)).$$

By Corollary 9.5, the element  $v_2^{32}$  persists to the  $E_4$ -page of the MASS for  $M(1, 4) \wedge DM(1)$ . By Proposition 2.1, our main theorem (Theorem 1.1) is a consequence of the following lemma.

**Lemma 11.1.** The element

$$v_2^{32} \in \text{Ext}_{A_*}^{32,224}(H(1, 4) \otimes DH(1))$$

cannot support a non-trivial  $d_r$  in the MASS for  $M(1, 4) \wedge DM(1)$  for  $r \geq 4$ .

*Proof.* We shall make use of the following tables. Table 11.1 depicts the algebraic  $tmf$ -resolution for  $\text{Ext}_{A_*}^{*,*}(H(1,4) \otimes DH(1))$  in the region where all possible targets of  $d_r(v_2^{32})$  can lie, for  $r \geq 4$ . Note that there are no non-zero elements in the algebraic  $tmf$ -resolution that can contribute to  $\text{Ext}_{A_*}^{s,191+s}(H(1,4) \otimes DH(1))$  for  $s > 40$ . Table 11.2 depicts a region of the algebraic  $tmf$ -resolution which maps to the region of Table 11.1 under  $g^6$ -multiplication. The notation in these tables is explained in Section 10. The subgroups labeled  $G_{191}$  and  $G_{71}$  are the subgroups generated by the contributions in the algebraic  $tmf$ -resolution labeled with a  $*$ .

| $s \setminus t - s$ | 190  | 191   |
|---------------------|--|---|
| 40                  | •  | ••  |
| 39                  | •  | •   |
| 38                  | •••<br>$g^6 b_{70} \circ \circ g^6 c_{70}$ | ••<br>$g^6 f_{71} \circ$  |
| 37                  | ••<br>◦<br>$g^6 a_{70} \odot$              | $v_2^8 k_{143} \bullet \bullet v_2^8 l_{143}$<br>$g^6 d_{71} \circ \circ g^6 e_{71}$<br>$g^6 b_{71} \odot \odot g^6 c_{71}$ |
| 36                  | ••<br>◊                                    | •<br>$g^6 a_{71} \odot$<br>$G_{191} *$  |

TABLE 11.1. The algebraic  $tmf$ -resolution for  $\text{Ext}_{A_*}^{*,*}(H(1,4) \otimes DH(1))$  in the vicinity of  $(t-s, s) = (191, 36)$

| $s \setminus t - s$ | 70                                | 71   |
|---------------------|-----------------------------------|--|
| 15                  | •                                 | •  |
| 14                  | ••<br>$b_{70} \circ \circ c_{70}$ | ••<br>$f_{71} \circ$   |
| 13                  | •<br>◦◦<br>$a_{70} \odot$         | $d_{71} \circ \circ \circ e_{71}$<br>$b_{71} \odot \odot c_{71}$ |
| 12                  | ◊<br>*                            | $a_{71} \odot \odot \odot \odot \odot$<br>$G_{71} *$             |

TABLE 11.2. The algebraic  $tmf$ -resolution for  $\text{Ext}_{A_*}^{*,*}(H(1,4) \otimes DH(1))$  in the vicinity of  $(t-s, s) = (71, 12)$

The element  $v_2^{32} \in \text{Ext}_{A(2)_*}^{32,224}(H(1,4))$  detects a non-trivial permanent cycle of order 2 in  $tmf_{192}M(1,4)$ . We deduce that the element

$$v_2^{32} \in \text{Ext}_{A(2)_*}(H(1,4) \otimes DH(1))$$

is a permanent cycle in the MASS for  $tmf \wedge M(1,4) \wedge DM(1)$ . Consider the map of MASSs

$$(11.1) \quad \begin{array}{ccc} \text{Ext}_{A_*}^{s,t}(H(1,4) \otimes DH(1)) & \Longrightarrow & \pi_{t-s}(M(1,4) \wedge DM(1)) \\ \downarrow & & \downarrow \\ \text{Ext}_{A(2)_*}^{s,t}(H(1,4) \otimes DH(1)) & \Longrightarrow & tmf_{t-s}(M(1,4) \wedge DM(1)) \end{array}$$

induced by the map

$$M(1, 4) \wedge DM(1) \rightarrow tmf \wedge M(1, 4) \wedge DM(1).$$

Because  $v_2^{32}$  is a permanent cycle in the MASS for  $tmf \wedge M(1, 4) \wedge DM(1)$ , we deduce that in the MASS for  $M(1, 4) \wedge DM(1)$ , the differential  $d_r(v_2^{32})$  cannot hit an element coming from  $\text{Ext}_{A(2)*}$  in the algebraic  $tmf$ -resolution (these elements are represented by a  $\bullet$  in Table 11.1). Thus the only possible targets for  $d_r(v_2^{32})$  in Table 11.1 are

$$(11.2) \quad g^6 a_{71}, g^6 b_{71}, g^6 c_{71}, g^6 d_{71}, g^6 e_{71}, g^6 f_{71}$$

or an element of the group  $G_{191}$ . We claim that none of these elements persist to detect a non-trivial element of the  $E_4$ -page of the MASS.

Each of the elements in (11.2) is in the image of multiplication by  $g^6$ . Because the groups  $G_{71}$  and  $G_{191}$  lie on the edge of the slope  $1/5$  vanishing line of Lemmas 7.4 and 7.5, each of the elements in  $G_{191}$  are of the form  $g^6 y$  for  $y \in G_{71}$ .

Suppose that  $x$  is a linear combination of the elements

$$(11.3) \quad a_{71}, b_{71}, c_{71}, d_{71}, e_{71}, f_{71}$$

and the elements in  $G_{71}$ . We must show that  $g^6 x$  cannot be the non-trivial image of  $d_r(v_2^{32})$  for  $r \geq 4$ .

If  $x$  is a  $d_r$ -cycle for  $r \leq 3$ , then  $x$  persists to  $E_4$ . Using the multiplicative structure of the MASS (Proposition 3.2) together with the fact that  $g^6 = 0$  in the  $E_4$ -page of the MASS for  $M(1, 4) \wedge DM(1, 4)$  (Theorem 10.1)<sup>1</sup>, we deduce that  $g^6 x$  is zero in  $E_4$ . It therefore cannot be a non-trivial target for  $d_r(v_2^{32})$ .

Suppose, however, that  $d_r(x)$  is non-trivial for some  $r \leq 3$ . Since differentials in the algebraic  $tmf$  resolution must increase filtration, we deduce that the only possible targets for  $d_r(x)$  are linear combinations of

$$a_{70}, b_{70}, c_{70}$$

and  $\bullet$ 's in Table 11.2 for which  $t - s = 70$  and  $s \geq 14$ . However, each of these  $\bullet$ 's map to non-trivial permanent cycles under the map of spectral sequences (11.1), and therefore cannot be the target of MASS differentials. The only remaining possibilities are

$$\text{Case (1):} \quad d_1(x) = a_{70},$$

$$\text{Case (2):} \quad d_2(x) = t_1 b_{70} + t_2 c_{70},$$

for  $(0, 0) \neq (t_1, t_2) \in \mathbb{F}_2 \oplus \mathbb{F}_2$ . Using Theorem 10.1 we see that in these cases we would respectively have:

$$\text{Case (1):} \quad d_1(g^6 x) = g^6 a_{70},$$

$$\text{Case (2):} \quad d_2(g^6 x) = t_1 g^6 b_{70} + t_2 g^6 c_{70}.$$

If we are in Case (1), we are done: the differential  $d_r(v_2^{32})$  cannot be detected by  $g^6 x$  because  $g^6 x$  does not persist to  $E_2$ . If we are in Case (2), however, we must verify that  $t_1 g^6 b_{70} + t_2 g^6 c_{70}$  is not in the image of a  $d_1$ -differential. The only possibility is

$$(11.4) \quad d_1(s_1 v_2^8 k_{143} + s_2 v_2^8 l_{143}) = t_1 g^6 b_{70} + t_2 g^6 c_{70}.$$

<sup>1</sup> This statement must be interpreted with care — Theorem 10.1 asserts that there is an element in  $E_2$  of the MASS for  $M(1, 4) \wedge DM(1, 4)$  which is detected by  $\tilde{g}^6$  in the algebraic  $tmf$ -resolution, and which is the target of a  $d_3$  in the MASS.

The algebraic  $tmf$ -resolution for  $\text{Ext}_{A_*}(H(1,4) \otimes DH(1))$  in the vicinity of the elements  $k_{143}$  and  $l_{143}$  is displayed below.

| $s \backslash t - s$ | 142 | 143                     |
|----------------------|-----|-------------------------|
| 30                   | •   | •                       |
| 29                   | ••  | $k_{143}$ • • $l_{143}$ |

We see that  $k_{143}$  and  $l_{143}$  must be  $d_1$ -cycles. By Proposition 4.3 and Lemma 5.7, we deduce that  $v_2^8 k_{143}$  and  $v_2^8 l_{143}$  must be  $d_1$ -cycles. Thus Possibility (11.4) cannot occur, and we deduce that in Case (2),  $d_2(g^6 x)$  does not vanish. We conclude that in Case (2),  $g^6 x$  cannot persist to  $E_4$  and therefore it cannot be the target of  $d_r(v_2^{32})$  for  $r \geq 4$ .  $\square$

## REFERENCES

- [Ada66] J. F. Adams, *On the groups  $J(X)$ . IV*, Topology **5** (1966), 21–71.
- [Bau08] Tilman Bauer, *Computation of the homotopy of the spectrum  $tmf$* , Groups, homotopy and configuration spaces (Tokyo 2005), Geometry and Topology Monographs, vol. 13, Geometry & Topology Publications, Coventry, 2008, pp. 11–40.
- [Beh07] Mark Behrens, *Some root invariants at the prime 2*, Proceedings of the Nishida Fest (Kinosaki 2003), Geom. Topol. Monogr., vol. 10, Geom. Topol. Publ., Coventry, 2007, pp. 1–40 (electronic).
- [BMMS86] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger,  *$H_\infty$  ring spectra and their applications*, Lecture Notes in Mathematics, vol. 1176, Springer-Verlag, Berlin, 1986.
- [BP04] Mark Behrens and Satya Pemmaraju, *On the existence of the self map  $v_2^9$  on the Smith-Toda complex  $V(1)$  at the prime 3*, Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic  $K$ -theory, Contemp. Math., vol. 346, Amer. Math. Soc., Providence, RI, 2004, pp. 9–49.
- [Bru93] Robert R. Bruner, *Ext in the nineties*, Algebraic topology (Oaxtepec, 1991), Contemp. Math., vol. 146, Amer. Math. Soc., Providence, RI, 1993, pp. 71–90.
- [DHS88] Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. I*, Ann. of Math. (2) **128** (1988), no. 2, 207–241.
- [DM81] Donald M. Davis and Mark Mahowald,  *$v_1$ - and  $v_2$ -periodicity in stable homotopy theory*, Amer. J. Math. **103** (1981), no. 4, 615–659.
- [DM82] ———, *Ext over the subalgebra  $A_2$  of the Steenrod algebra for stunted projective spaces*, Current trends in algebraic topology, Part 1 (London, Ont., 1981), CMS Conf. Proc., vol. 2, Amer. Math. Soc., Providence, RI, 1982, pp. 297–342.
- [HG94] M. J. Hopkins and B. H. Gross, *The rigid analytic period mapping, Lubin-Tate space, and stable homotopy theory*, Bull. Amer. Math. Soc. (N.S.) **30** (1994), no. 1, 76–86.
- [HM] Michael J. Hopkins and Mark Mahowald, *From elliptic curves to homotopy theory*, <http://hopf.math.purdue.edu/>.
- [HS98] Michael J. Hopkins and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. II*, Ann. of Math. (2) **148** (1998), no. 1, 1–49.
- [Mah81] Mark Mahowald, *The primary  $v_2$ -periodic family*, Math. Z. **177** (1981), no. 3, 381–393.
- [MR99] Mark Mahowald and Charles Rezk, *Brown-Comenetz duality and the Adams spectral sequence*, Amer. J. Math. **121** (1999), no. 6, 1153–1177.
- [Nak72] Osamu Nakamura, *On the squaring operations in the May spectral sequence*, Mem. Fac. Sci. Kyushu Univ. Ser. A **26** (1972), no. 2, 293–308.
- [Rav86] Douglas C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121, Academic Press Inc., Orlando, FL, 1986.
- [Rav92] ———, *Nilpotence and periodicity in stable homotopy theory*, Annals of Mathematics Studies, vol. 128, Princeton University Press, Princeton, NJ, 1992, Appendix C by Jeff Smith.
- [Smi70] Larry Smith, *On realizing complex bordism modules. Applications to the stable homotopy of spheres*, Amer. J. Math. **92** (1970), 793–856.

- [Tan70] Martin C. Tangora, *On the cohomology of the Steenrod algebra*, Math. Z. **116** (1970), 18–64.
- [Tod71] Hirosi Toda, *On spectra realizing exterior parts of the Steenrod algebra*, Topology **10** (1971), 53–65.
- [Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.

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